Finding Antiderivatives by u Substitution

To understand how the substitution works in computing antiderivatives, we need to first make sure we understand how the chain rule works. Let's start by taking the derivative of

$$y = \left(x^2 + 2x\right)^7$$

To apply the chain rule to this function, we identify the inside part, g(x), of the right side as

$$g(x) = x^2 + 2x$$

and the outside part, f(x), as

$$f(x) = x^7$$

This means that the function is being written as a composition in the form $f(g(x))$. The derivatives of these functions are

$$g(x) = x^2 + 2x \qquad \rightarrow \qquad g'(x) = 2x + 2$$

$$f(x) = x^7 \qquad \rightarrow \qquad f'(x) = 7x^6$$

This results in the derivative

$$\frac{dy}{dx} = \underbrace{7\left(x^2 + 2x\right)^6}_{f'(g(x))} \underbrace{\left(2x + 2\right)}_{g'(x)}$$

The corresponding antiderivative would be

$$\int 7\left(x^2 + 2x\right)^6 \left(2x + 2\right) dx = \left(x^2 + 2x\right)^7 + C$$

The derivative and antiderivative are opposite processes of each other:

$$\left(x^2 + 2z\right)^7 \quad \underset{\xleftarrow[\text{u substitution}]{}}{\xrightarrow{\text{chain rule}}} \quad 7\left(x^2 + 2x\right)^6 \left(2x + 2\right)$$

The opposite process of the chain rule is called u substitution. In this antiderivative technique, the inside function g(x) is called u and is used to simplify the integrand. Let's look at how this is done. We'll find the antiderivative

$$\int 7\left(x^2 + 2x\right)^6 \left(2x + 2\right) dx$$

Identify the inside function as $u = x^2 + 2x$. The derivative of the inside function is $\frac{du}{dx} = 2x + 2$. We can find each of these functions in the integrand:

$$\int 7\underbrace{\left(x^2 + 2x\right)}_{u}^6 \underbrace{\left(2x + 2\right)}_{\frac{du}{dx}} dx = \int 7u^6 \cdot \frac{du}{\cancel{dx}} \cdot \cancel{dx} = \int 7u^6 \, du$$

Notice that this is simply the antiderivative of the outside function, $f'(u)$. We can evaluate this antiderivative with the power rule for antiderivatives,

$$\int 7u^6 \, du = u^7 + C$$

Since the original variable in the p[problem was x, we need to get back to that variable using $u = x^2 + 2x$. This means the antiderivative is $\left(x^2 + 2x\right)^7 + C$ or

$$\int 7\left(x^2 + 2x\right)^6 \left(2x + 2\right) dx = \left(x^2 + 2x\right)^7 + C$$

A Slightly More Complicated Example

In the example above, the $\frac{du}{dx}$ was easy to find in the integrand. What if the integrand does not match up with $\frac{du}{dx} = 2x + 2$ perfectly? Suppose we want to find the antiderivative

$$\int 7\left(x^2 + 2x\right)^6 \left(x + 1\right) dx$$

Instead of a factor of $(2x + 2)$ in the integrand, we have a half of this factor or $(x + 1)$. The easiest way to compensate for this lack of a factor of 2 is to put it in. However, then we need to balance this factor out by multiplying by $\frac{1}{2}$. Typically we do this in front of the antiderivative symbol:

$$\int 7\left(x^2 + 2x\right)^6 \left(x + 1\right) = \tfrac{1}{2} \int 7\left(x^2 + 2x\right)^6 \left(2x + 2\right) dx$$

The factors in red are the same as $(x + 1)$ since $\frac{1}{2}\left(2x + 2\right) = x + 1$. The advantage here is that now we can see the factor corresponding to $\frac{du}{dx}$ in the integrand. Letting $u = x^2 + 2x$ and $\frac{du}{dx} = 2x + 2$ allow us to rewrite the right hand side of the equation above as

$$\tfrac{1}{2}\int 7\underbrace{\left(x^2+2x\right)}_{u}^{6}\underbrace{\left(2x+2\right)}_{\frac{du}{dx}}\,dx = \tfrac{1}{2}\int 7u^6\cdot\frac{du}{\cancel{dx}}\cdot\cancel{dx}$$

$$= \tfrac{1}{2}\int 7u^6\;du$$

The antiderivative is found with the power rule as $\tfrac{1}{2}u^7 + C$ so the final solution is

$$\int 7\left(x^2+2x\right)^6\left(x+1\right)dx = \tfrac{1}{2}\left(x^2+2x\right)^7 + C$$

The $\frac{1}{2}$ in the antiderivative balances out the doubling we needed to do to introduce the correct derivative of u.