Question 3: What are the derivatives of some basic functions (constant, linear, power, polynomial, exponential, and logarithmic)?

There are many symbols used in calculus that indicate derivative or that a derivative must be taken.

> The derivative of $y = f(x)$ may be symbolized in any of the following ways:
>
> $$f'(x), \quad \frac{dy}{dx}, \quad \frac{d}{dx}[f(x)] \text{ or } D_x[f(x)]$$

For instance, the derivative of the function $f(x) = 2x - 7$ is $f'(x) = 2$. Using the other notations for derivative, we could also write

$$\frac{dy}{dx} = 2 \qquad \frac{d}{dx}[2x-7] = 2 \qquad D_x[2x-7] = 2$$

If the function uses a different independent variable, we adjust the notation to reflect this variable. The function $g(t) = t^2 + 3t + 7$ has a derivative $g'(t) = 2t + 3$. This could also be written as

$$\frac{dy}{dt} = 2t+3 \qquad \frac{d}{dt}\left[t^2+3t+7\right] = 2t+3 \qquad D_t\left[t^2+3t+7\right] = 2t+3$$

In many business applications the independent variable and dependent variables may have names to reflect what they represent. In a demand function, $P = D(Q)$, the variable P represents the unit price of some item and Q represents the number of items sold at that price. If the variables are related by $P = 100 - 0.25Q^2$, we can write the derivative as

$$D'(Q) = -0.50Q$$

$$\frac{dP}{dQ} = -0.50Q$$

$$\frac{d}{dQ}\left[100 - 0.25Q^2\right] = -0.50Q$$

$$D_Q\left[100 - 0.25Q^2\right] = -0.50Q$$

In this text, we will use each of these notations to help familiarize you with each one. In most non-mathematics texts, the author may have a favorite notation and use that notation exclusively.

The process of taking the derivative of a function is called differentiation. Additionally, the term differentiate indicates that the derivative should be taken.

The derivatives of many functions follow patterns. If we know these patterns, we can avoid computing derivatives from the definition directly. The simplest of these patterns is the Constant Rule.

The Constant Rule for Derivatives

If $f(x) = c$, where c is a real number, then the derivative of $f(x)$ is

$$f'(x) = 0$$

In simple terms, a constant function does not change so the instantaneous rate of change should be equal to zero.

Example 5 Find the Derivative of a Constant Function

Find each of the derivatives below.

a. $\frac{dF}{dx}$ if $F(x) = 12$

Solution Since the derivative of a constant is equal to zero,

$$\frac{dF}{dx} = \frac{d}{dx}[12] = 0$$

b. $\frac{d}{dt}\left[\pi^2\right]$

Solution On the surface it might appear that the constant rule for derivatives does not apply because of the power. But the derivative is being taken with respect to t, not π. The symbol π is constant with respect to t so

$$\frac{d}{dt}\left[\pi^2\right] = 0$$

Linear functions are commonly encountered in business and economics. Since they rise or fall at a constant rate, the derivative of a linear function is simply the slope of the linear function.

Derivative of a Linear Function

If $f(x) = ax + b$, where a and b are real numbers, then the derivative of $f(x)$ is

$$f'(x) = a$$

In this case, the slope of the linear function is a, the coefficient on the variable x.

Example 6 Find the Derivative of a Linear Function

Find each of the derivatives below.

a. $\frac{dy}{dx}$ if $y = 2x - 3$

Solution The derivative of a linear function is equal to the coefficient on the variable,

$$\frac{dy}{dx} = \frac{d}{dx}\left[2x - 3\right] = 2$$

b. $D_t\left[\frac{1}{2}t + \frac{3}{2}\right]$

Solution The variable in this linear function is t, but the rule for derivatives of linear functions still applies,

$$D_t\left[\frac{1}{2}t + \frac{3}{2}\right] = \frac{1}{2}$$

One of the most useful derivative formulas is the Power Rule for Derivatives.

The Power Rule for Derivatives

If $f(x) = x^n$, where n is any real number, then the derivative of $f(x)$ is

$$f'(x) = n\,x^{n-1}$$

The Power Rule for Derivatives is easy to apply to functions where the function has the form of a variable raised to a constant. In addition, any function such as a root function that can be converted to a variable raised to constant can also be differentiated with the Power Rule for Derivatives.

Example 7 Find the Derivative of a Power Function

Find each of the derivatives below.

a. $\frac{d}{dx}\left[x^3\right]$

Solution Apply the Power Rule for Derivatives to give

$$\begin{aligned} \frac{d}{dx}\left[x^3\right] &= 3x^{3-1} \\ &= 3x^2 \end{aligned}$$

So $\frac{d}{dx}\left[x^3\right] = 3x^2$.

b. $\frac{dy}{dt}$ if $y = t^{1.5}$

Solution Although the variable in this function is not x, we can still apply the Power Rule in the form $\frac{d}{dt}\left[t^n\right] = nt^{n-1}$. The Power Rule applies to power functions with any type of number, such as a decimal,in the power:

$$\begin{aligned} \frac{dy}{dt} &= \frac{d}{dt}\left[t^{1.5}\right] \\ &= 1.5t^{1.5-1} \\ &= 1.5t^{0.5} \end{aligned}$$

The derivative is $\frac{dy}{dt} = 1.5t^{0.5}$.

c. $r'(x)$ if $r(x) = \sqrt{x}$

Solution Although this function does not look like a power function, we can rewrite it as $r(x) = x^{\frac{1}{2}}$ and apply the Power Rule for Derivatives,

$$\begin{aligned} r'(x) &= \frac{d}{dx}\left[x^{\frac{1}{2}}\right] \\ &= \frac{1}{2}x^{\frac{1}{2}-1} \\ &= \frac{1}{2}x^{-\frac{1}{2}} \\ &= \frac{1}{2}\frac{1}{\sqrt{x}} \end{aligned}$$

Use $x^{-\frac{1}{2}} = \frac{1}{x^{\frac{1}{2}}}$ and $x^{\frac{1}{2}} = \sqrt{x}$

The derivative is $r'(x) = \frac{1}{2\sqrt{x}}$.

Example 8 Find the Derivative of a Power Function

Find each of the derivatives below.

a. $\frac{d}{dx}\left[\frac{1}{x^5}\right]$

Solution To be able to use the Power Rule for Derivatives, we need to take advantage of the fact that $\frac{1}{x^5} = x^{-5}$. Apply the Power Rule to give

$$\frac{d}{dx}\left[\frac{1}{x^5}\right] = \frac{d}{dx}\left[x^{-5}\right]$$

$$= -5x^{-5-1}$$

$$= -5x^{-6}$$

$$= \frac{-5}{x^6} \qquad \text{Use } x^{-6} = \frac{1}{x^6}$$

So $\frac{d}{dx}\left[\frac{1}{x^5}\right] = \frac{-5}{x^6}$

b. $D'(Q)$ if $D(Q) = \frac{1}{\sqrt[3]{Q}}$

Solution Rewrite the function using a negative fraction exponent,

$\frac{1}{\sqrt[3]{Q}} = Q^{-\frac{1}{3}}$, and apply the power rule,

$$D'(Q) = \frac{d}{dQ}\left[Q^{-\frac{1}{3}}\right]$$

$$= -\frac{1}{3}Q^{-\frac{1}{3}-1}$$

$$= -\frac{1}{3}Q^{-\frac{4}{3}}$$

$$= -\frac{1}{3}\frac{1}{\sqrt[3]{Q^4}} \qquad \text{Use } Q^{-\frac{4}{3}} = \frac{1}{Q^{\frac{4}{3}}} = \frac{1}{\sqrt[3]{Q^4}}$$

The derivative is $D'(Q) = -\frac{1}{3\sqrt[3]{Q^4}}$.

Several rules must be used to differentiate polynomial functions. The first rule helps us to find the derivative of products where one of the factors is a constant.

Derivatives of a Constant Times a Function

If a is a real number and $f(x)$ is a differentiable function, then

$$\frac{d}{dx}\left[a\, f(x)\right] = a\ f'(x)$$

This rule tells us that the derivative of a constant times a function is equal to the constant times the derivative of the function.

Example 9 Find the Derivative of a Power Function Multiplied by a Constant

Find each of the derivatives below.

a. $\frac{d}{dt}\left[5t^{10}\right]$

Solution Use the Product with a Constant Rule and the Power Rule to calculate the derivative,

$$\begin{aligned}\frac{d}{dt}\left[5t^{10}\right] &= 5\cdot\frac{d}{dt}\left[t^{10}\right] && \text{Use the Constant Times a Function Rule}\\ &= 5(10t^{9}) && \text{Use the Power Rule}\\ &= 50t^{9}\end{aligned}$$

So $\frac{d}{dt}\left[5t^{10}\right] = 50t^{9}$.

b. $R'(Q)$ if $R(Q) = 32Q^{0.765}$

Solution Use the Product with a Constant Rule and the Power Rule to calculate the derivative,

$$R'(Q) = \frac{d}{dQ}\left[32Q^{0.765}\right]$$

$$= 32 \cdot \frac{d}{dQ}\left[Q^{0.765}\right] \qquad \text{Use the Constant Times a Function Rule}$$

$$= 32\left(0.765 \cdot Q^{-0.235}\right) \qquad \text{Use the Power Rule}$$

$$= 24.48Q^{-0.235}$$

The derivative is $R'(Q) = 24.48Q^{-0.235}$.

The next rule applies to differentiating sums or differences of functions.

Sum or Difference Rule for Derivatives

If $f(x)$ and $g(x)$ are differentiable functions, then

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = f'(x) \pm g'(x)$$

The derivative of a sum or difference of functions is equal to the sum or difference of the derivatives of the functions. This rule can be extended to take the derivative of any number of functions that are added or subtracted. If we combine this rule with rule for taking derivatives of products with a constant, we can take the derivative of any polynomial.

Example 10 Find the Derivative of a Polynomial

Find each of the derivatives below.

a. $D_x\left[3x^3+1.5x^2-2x\right]$

Solution The derivative of a sum or difference is the sum or difference of the derivatives,

$$\begin{aligned} D_x\left[3x^3+1.5x^2-2x\right] &= D_x\left[3x^3\right]+D_x\left[1.5x^2\right]-D_x\left[2x\right] \\ &= 3D_x\left[x^3\right]+1.5D_x\left[x^2\right]-2D_x\left[x\right] \\ &= 3\left(3x^2\right) + 1.5\left(2x\right) - 2\left(1\right) \\ &= 9x^2+3x-2 \end{aligned}$$

Use the Sum / Difference Rule with the Constant Times a Function Rule

Use the Power Rule

So $D_x\left[3x^3+1.5x^2-2x\right]=9x^2+3x-2$

b. $\frac{d}{dz}\left[\frac{1}{4}z^5-\sqrt{2}z^3+e^2\right]$

Solution Break the derivative of a polynomial into the sum and difference of the derivatives of the terms,

$$\begin{aligned} \frac{d}{dz}\left[\tfrac{1}{4}z^5-\sqrt{2}z^3+e^2\right] &= \frac{d}{dz}\left[\tfrac{1}{4}z^5\right]-\frac{d}{dz}\left[\sqrt{2}z^3\right]+\frac{d}{dz}\left[e^2\right] \\ &= \tfrac{1}{4}\frac{d}{dz}\left[z^5\right]-\sqrt{2}\frac{d}{dz}\left[z^3\right]+\frac{d}{dz}\left[e^2\right] \\ &= \tfrac{1}{4}\left(5z^4\right) - \sqrt{2}\left(3z^2\right) + 0 \end{aligned}$$

Use the Sum / Difference Rule with the Constant Times a Function Rule

Use Power Rule on the first two terms

Take note of the third term in this polynomial. Even though the third term looks like a power function, it is a constant since e is a constant. Therefore the derivative of the third term is zero.

Putting all of these terms together gives

$$\frac{d}{dz}\left[\tfrac{1}{4}z^5-\sqrt{2}z^3+e^2\right]=\tfrac{5}{4}z^4-3\sqrt{2}\,z^2\,.$$

As we saw in an earlier question, the derivative of the exponential function e^x is easy to remember.

If $f(x)=e^x$, the the derivative of $f(x)$ is

$$f'(x)=e^x$$

In other words, the derivative of e^x is e^x. Exponential functions with a base other than e are found with another simple formula.

If $f(x)=a^x$, where a is a positive real number, then the derivative of $f(x)$ is

$$f'(x)=(\ln a)\,a^x$$

Example 11 Find the Derivative of Exponential Functions

Find each of the derivatives.

a. $f'(x)$ if $f(x)=17.1e^x$

Solution Use the Product with a Constant Rule to help take the derivative of the exponential function,

$$f'(x) = \frac{d}{dx}\left[17.1e^x\right]$$

$$= 17.1\frac{d}{dx}\left[e^x\right]$$ Use the Constant Times a Function Rule

$$= 17.1e^x$$

The derivative is $f'(x) = 17.1e^x$

b. $S'(t)$ if $S(t) = 17,000\left(\frac{1}{2}\right)^t$

Solution Use the Constant Rule to help take the derivative of the exponential function,

$$S'(t) = \frac{d}{dt}\left[17,000\left(\tfrac{1}{2}\right)^t\right]$$

$$= 17,000\frac{d}{dt}\left[\left(\tfrac{1}{2}\right)^t\right]$$ Use Constant Times a Function Rule

$$= 17,000\ln\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)^t$$

The derivative is $S'(t) = 17,000\ln\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^t$.

c. $\frac{d}{dx}\left[3^{2x}\right]$

Solution Since the power in the exponential is $2x$ instead of x, we cannot use the derivative rule for exponential functions directly. However, we can rewrite this exponential function as $3^{2x} = \left(3^2\right)^x = 9^x$. In this format, we can apply the derivative rule for exponential functions to yield

$$\frac{d}{dx}\left[3^{2x}\right] = \frac{d}{dx}\left[9^x\right]$$

$$= \ln(9)\,9^x$$

So $\frac{d}{dx}\left[3^{2x}\right]=\ln(9)\,9^x$.

The last type of function we'll learn to differentiate is a logarithmic function.

If $f(x)=\log_a(x)$, where a is a positive real number, then the derivative of $f(x)$ is

$$\frac{d}{dx}\left[\log_a(x)\right]=\frac{1}{\ln(a)}\cdot\frac{1}{x}$$

In the case where the base of the logarithm is e, this relationship simplifies considerably because:

$$\frac{d}{dx}\left[\log_e(x)\right]=\frac{1}{\ln(e)}\cdot\frac{1}{x}=\frac{1}{x}$$

Since $\log_e(x)$ is the same as $\ln(x)$, we can write another basic derivative.

If $f(x)=\ln(x)$, the derivative of $f(x)$ is

$$\frac{d}{dx}\left[\ln(x)\right]=\frac{1}{x}$$

Example 12 Find the Derivative of Logarithmic Functions

Find each of the derivatives.

a. $g'(x)$ if $g(x)=153\ln(x)$

Solution For this function, we can use the Product with a Constant Rule prior to taking the derivative of the logarithm to give

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left[153\ln(x)\right] \\ &= 153\frac{d}{dx}\left[\ln(x)\right] \\ &= 153\left(\frac{1}{x}\right) \end{aligned}$$

The derivative is $g'(x) = \dfrac{153}{x}$.

b. $D_t\left[\ln\left(t^2\right)\right]$

Solution In this part, there is a t^2 instead of a t in the logarithm as required by the derivative rule for logarithms. However, the properties of logarithms allow the power in the logarithm to be moved outside of the logarithm, $\ln\left(t^2\right) = 2\ln(t)$. Now we can apply the derivative rule for logarithms,

$$\begin{aligned} D_t\left[\ln\left(t^2\right)\right] &= D_t\left[2\ln(t)\right] \\ &= 2\,D_t\left[\ln(t)\right] \\ &= 2\left(\frac{1}{t}\right) \end{aligned}$$

Use the rules for simplifying logarithms, $\log_a\left(x^b\right) = b\log_a(x)$

Use the Constant Times a Function Rule

So $D_t\left[\ln\left(t^2\right)\right] = \dfrac{2}{t}$

c. $\dfrac{d}{dz}\left[\log_2(9z)\right]$

Solution Use the product property of logarithms to write

$$\log_2(9z) = \log_2(9) + \log_2(z)$$

The derivative can now be worked out with the derivative rule for logarithms,

$$\frac{d}{dz}\left[\log_2(9z)\right] = \frac{d}{dz}\left[\log_2(9) + \log_2(z)\right]$$

Use Product Property of Logarithms, $\log_a(xy) = \log_a(x) + \log_a(y)$

$$= \frac{d}{dz}\left[\log_2(9)\right] + \frac{d}{dz}\left[\log_2(z)\right]$$

Use Sum Rule for Derivatives

$$= \quad 0 \quad + \quad \frac{1}{\ln(2)}\frac{1}{z}$$

In the first term, the derivative of a constant is zero

So $\frac{d}{dz}\left[\log_2(9z)\right] = \frac{1}{\ln(2)}\frac{1}{z}$