Question 2: How do you find the derivative of a quotient of two functions?

A quotient of two functions is where two functions are divided. If we can write a function as a quotient of two function $u(x)$ and $v(x)$, the Quotient Rule for Derivatives is used to take the derivative.

The Quotient Rule for Derivatives

If $u(x)$ and $v(x)$ are differentiable functions, then

$$\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{v(x)u'(x) - u(x)v'(x)}{\left(v(x)\right)^2}$$

The Quotient Rule for Derivatives is easier to remember in abbreviated form,

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\,u' - u\,v'}{v^2}$$

Example 3 Find the Derivative of a Quotient

Suppose $h(x) = \dfrac{2x^2 + 6x + 52}{x}$.

a. Use the Quotient Rule for Derivatives to find $h'(x)$.

Solution To apply the Quotient Rule for Derivatives, we must identify the numerator u and the denominator v of the function $h(x) = \dfrac{2x^2 + 6x + 52}{x}$. In addition, we need the derivatives of these pieces,

$$u = 2x^2 + 6x + 52 \quad \rightarrow \quad u' = 4x + 6$$

$$v = x \quad \rightarrow \quad v' = 1$$

The Quotient Rule for Derivatives yields

$$h'(x) = \frac{\overbrace{(x)(4x+6)}^{vu'} - \overbrace{(2x^2+6x+52)(1)}^{uv'}}{\underbrace{x^2}_{v^2}}$$

To simplify this derivative, remove the parentheses and combine like terms in the numerator,

$$h'(x) = \frac{(x)(4x+6) - (2x^2+6x+52)(1)}{x^2}$$

$$= \frac{4x^2+6x-2x^2-6x-52}{x^2}$$ Remove the parentheses and multiply

$$= \frac{2x^2-52}{x^2}$$ Combine like terms in the numerator

b. Divide the trinomial in the numerator by the monomial in the denominator and then take the derivative to find $h'(x)$.

Solution When dividing any polynomial by a monomial, we can simplify the fraction by dividing each term in the numerator by the monomial,

$$h(x) = \frac{2x^2}{x} + \frac{6x}{x} + \frac{52}{x}$$ Divide each term by the monomial in the denominator

$$= 2x + 6 + 52x^{-1}$$ Simplify each term

Each term is now a constant or power function and easy to take the derivative of. Several derivative rules are used:

$$h'(x) = \frac{d}{dx}\left[2x+6+52x^{-1}\right]$$

$$= 2\frac{d}{dx}[x] + \frac{d}{dx}[6] + 52\frac{d}{dx}\left[x^{-1}\right]$$ Use the Sum Rule as well as the Product with a Constant Rule.

$$= 2(1) + 0 + 52\left(-1x^{-2}\right)$$ Use the Power Rule for Derivatives and the fact that the derivative of a constant is zero.

$$= 2 - 52x^{-2}$$

The derivative is $h'(x) = 2 - 52x^{-2}$. This derivative can also be written as

$$h'(x) = 2 - \frac{52}{x^2} \quad \text{or} \quad h'(x) = \frac{2x^2 - 52}{x^2}$$

by converting the negative exponent to a positive exponent.

The derivative matches the derivative from part a. For this particular quotient, the derivative can be taken with the quotient rule for derivatives or the quotient can be simplified and then the derivative taken resulting in the same derivative.

In the next example we will not be able to simplify the quotient to avoid the use of the quotient rule. For quotients like this, we must identify the numerator u and the denominator v and apply the Quotient Rule to find the derivative.

Example 4 Find the Derivative of a Quotient

Find the derivatives indicated in each part using the Quotient Rule.

a. $\frac{dy}{dx}$ if $y = \frac{e^x}{x}$

Solution The numerator u, denominator v and their corresponding derivatives are

$$u = e^x \quad \rightarrow \quad u' = e^x$$
$$v = x \quad \rightarrow \quad v' = 1$$

The derivative $\frac{dy}{dx}$ is computed using the Quotient Rule,

$$\frac{dy}{dx} = \frac{x \cdot e^x - e^x \cdot 1}{x^2}$$

Use $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$

$$= \frac{e^x\left(x-1\right)}{x^2}$$

Factor e^x from each term in the numerator

The derivative is $\frac{dy}{dx} = \frac{e^x\left(x-1\right)}{x^2}$.

b. $D_y\left[\frac{-y^3+4y^2-5}{y^2+1}\right]$

Solution The numerator u, denominator v and their corresponding derivatives are

$$u = -y^3+4y^2-5 \quad \rightarrow \quad u' = -3y^2+8y$$

$$v = y^2+1 \quad \rightarrow \quad v' = 2y$$

The quotient rule for derivatives yields

$$D_y\left[\frac{-y^3+4y^2-5}{y^2+1}\right] = \frac{(y^2+1)(-3y^2+8y)-(-y^3+4y^2-5)(2y)}{\left(y^2+1\right)^2}$$

The numerator, $(y^2+1)(-3y^2+8y)-(-y^3+4y^2-5)(2y)$, can be simplified by multiplying the factors and combining like terms,

$$D_y\left[\frac{-y^3+4y^2-5}{y^2+1}\right] = \frac{-3y^4+8y^3-3y^2+8y+2y^4-8y^3+10y}{\left(y^2+1\right)^2}$$

$$= \frac{-y^4-3y^2+18y}{\left(y^2+1\right)^2}$$

The derivative is $D_y\left[\frac{-y^3+4y^2-5}{y^2+1}\right] = \frac{-y^4-3y^2+18y}{\left(y^2+1\right)^2}$.