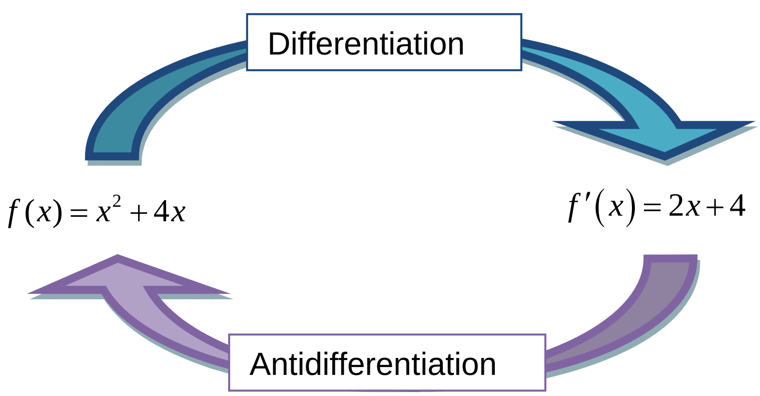
Question 1: What is an antiderivative?

Suppose we have the function $f(x) = x^2 + 4x$. Using the Power and Sum Rules for Derivatives, we can easily take the derivative of this function.

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\left[x^2 + 4x\right] \\
&= \frac{d}{dx}\left[x^2\right] + \frac{d}{dx}[4x] \\
&= 2x + 4
\end{aligned}
$$

In this context, we say that the derivative of $f(x) = x^2 + 4x$ is $f'(x) = 2x + 4$. This process is called differentiation.

Taking an antiderivative reverses this process. Starting from the derivative, we calculate the antiderivative. We would say that the antiderivative of $f'(x) = 2x + 4$ is $f(x) = x^2 + 4x$. Reversing differentiation is called antidifferentiation.



Using our knowledge of derivatives, we may deduce many antiderivatives. One of the most useful tools is the Power Rule for Derivatives. This rules states that the derivative of x^n is $n x^{n-1}$. In English, the power becomes the coefficient and the power is reduced by one. In reversing this process, we would expect the power to increase by one and the coefficient to be divided by some factor. The antiderivative of $2x$ is x^2. In this case the power on the x has been increased by one to two and the coefficient has been divided by 2. Initially, you might apply this strategy somewhat haphazardly and make some guesses. When in doubt, we can always check an antiderivative by taking its derivative. In other words, by taking the derivative of x^2 we insure that it is $2x$.

Example 1 Find an Antiderivative

Find an antiderivative of each expression.

a. An antiderivative of $f'(x) = 5$.

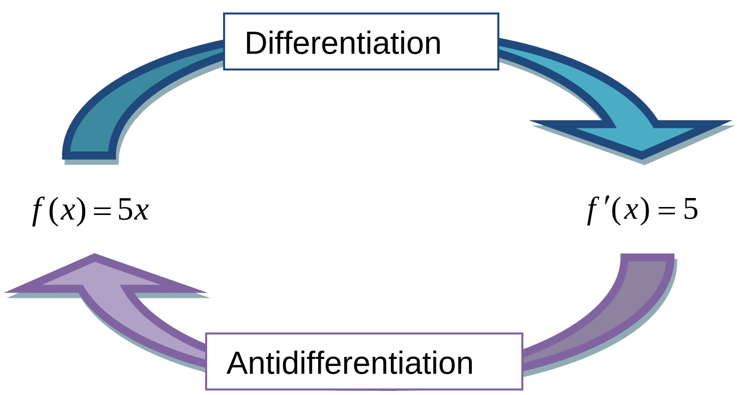
Solution To answer this question, we need to find the function $f(x)$ whose derivative is 5. Derivatives yield constants when the variable in the expression is raised to the first power. For instance,

$$\frac{d}{dx}[4x] = 4$$

To get a derivative to equal 5, we would need to start with $5x$ or

$$\frac{d}{dx}[5x] = 5$$

This means an antiderivative of 5 is $5x$.



b. An antiderivative of $R'(p) = 4p^3$.

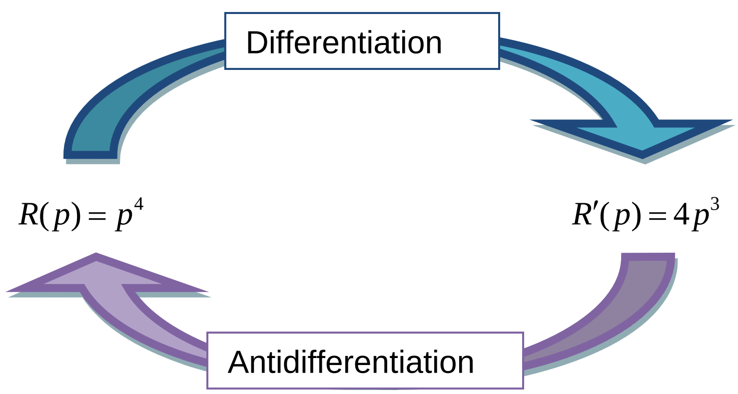
Solution For a function to be an antiderivative of $4p^3$, its derivative must equal $4p^3$. Symbollically, this can be written as

$$\frac{d}{dp}[?] = 4p^3$$

We know the Power Rule for Derivatives reduces the power on the variable by one, so the antiderivative must have a power of four on it. Let's try p^4. To see if this is correct, take its derivative to give

$$\frac{d}{dp}\left[p^4\right] = 4p^3$$

Since this is true, $R(p) = p^4$ is an antiderivative of $R'(p) = 4p^3$.



c. An antiderivative of $g'(x) = 10x^4$.

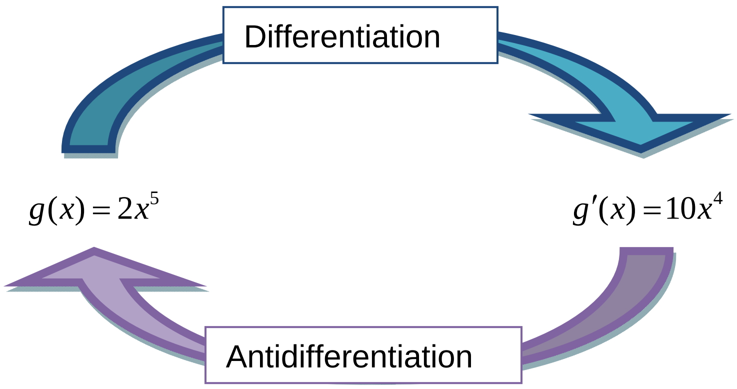
Solution Using the strategy of adding one to the power suggest trying an antiderivative of x^5. Check this by taking the antiderivative:

$$\frac{d}{dx}\left[x^5\right] = 5x^4$$

The correct antiderivative would have a derivative twice as big, $10x^4$. This suggests trying an antiderivative of $2x^5$. In this case, the derivative is

$$\frac{d}{dx}\left[2x^5\right] = 10x^4$$

Since this is the correct derivative, $2x^5$ is an antiderivative of $10x^4$.



In the last example, you may have noticed a strange use of "an" as in "$2x^5$ is an antiderivative of $10x^4$". This was not an accident. Any function whose derivative is $10x^4$ is an antiderivative of $10x^4$. There are many functions that meet this requirement.

$$\frac{d}{dx}\left[2x^5 + 1\right] = 10x^4$$

$$\frac{d}{dx}\left[2x^5 + 12\right] = 10x^4$$

$$\frac{d}{dx}\left[2x^5 - 5\right] = 10x^4$$

The functions $2x^5 + 1$, $2x^5 + 12$, and $2x^5 - 5$ are all antiderivatives of $10x^4$ since there derivatives are all equal to $10x^4$. In each case, the different constant on the end has no effect on the derivative since the derivative of a constant is zero. Any constant could be in the function and the derivative would still equal $10x^4$.

To account for these constants, we say that the antiderivative of $g'(x) = 10x^4$ is $g(x) = 2x^5 + C$, where C is an arbitrary constant. In general, we represent all of these antiderivatives using a letter like C.

Example 2 Find the Antiderivative

Find the antiderivative of $h'(x) = 100x^{99} + 25$.

Solution We need to find a function $h(x)$ whose derivative is equal to $h'(x) = 100x^{99} + 25$. Using the rules for derivatives, we can deduce one antiderivative $h(x) = x^{100} + 25x$. Since adding an arbitrary constant results in the exact same derivative, we can represent all antiderivatives of $h'(x)$ as $h(x) = x^{100} + 25x + C$.

Any letter can be used to describe the arbitrary constant as long as we identify it as such. In some situations, the letter C may be confusing. In particular, cost functions are normally named with the letter C. This might conflict with an arbitrary constant called C. Feel free to use a different letter for the constant in these situations.

The antiderivative is symbolized using a symbol called an integral. For Example 2, we would use it by writing

$$\int \left(100x^{99} + 25\right) dx = x^{100} + 25x + C$$

The integral of   with respect to x

This is read "the integral of $100x^{99} + 25$ with respect to x is $x^{100} + 25x + C$". The integral $\int$ indicates the antiderivative and the dx indicates the variable. The expression between the integral and the dx is called the integrand. As a group, the integral, integrand, and variable indicator are called the indefinite integral.

It is very important to know what the variable is in an antiderivative. In the next example, we find the integral of the same integrand, but with respect to two different variables.

Example 3 Find the Integral

Find each integral.

a. $\int 2xy\,dx$

Solution The symbol dx indicates that the variable in this antiderivative is x. The y is treated as a constant. We must find a function such that

$$\frac{d}{dx}[?] = 2xy$$

From earlier examples, we know an antiderivative of $2x$ is x^2. Treating y as a constant, we try

$$\frac{d}{dx}\left[x^2 y\right] = y \cdot \frac{d}{dx}\left[x^2\right] = y \cdot 2x$$

This tells us that

$$\int 2xy\,dx = x^2 y + C$$

b. $\int 2xy\,dy$

Solution The symbol dy indicates that y is the variable so x is treated as a constant. An antiderivative with respect to y is a fucntion such that

$$\frac{d}{dy}[?] = 2xy$$

Taking the derivative with respect to y means the power on the y must be increased by one so that

$$\frac{d}{dy}\left[xy^2\right] = x \cdot \frac{d}{dy}\left[y^2\right] = x \cdot 2y$$

Since this is the correct derivative,

$$\int 2xy\,dy = xy^2 + C$$

Compared to part a, the power on the variable y is raised by one instead of the power on the x.

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