Question 4: How do we find the value of the arbitrary constant?

The antiderivative may be used to find a function from its derivative. If we know the derivative of a function is

$$f'(x) = 30x^2 + 16x$$

then the function must be

$$\begin{aligned} f(x) &= \int f'(x)\,dx \\ &= \int \left(30x^2 + 16x\right) dx \\ &= 10x^3 + 8x^2 + C \end{aligned}$$

We can check this function by taking its derivative,

$$f'(x) = \frac{d}{dx}\left[10x^3 + 8x^2 + C\right] = 30x^2 + 16x$$

Each value of C defines a different function. This means there is an infinite number of functions whose derivative is $f'(x) = 30x^2 + 16x$. To find a unique value for C, we need some information about the function $f(x)$. If we are given the fact that $f(1) = 33$, we can use this information to find C. Substitute $x = 1$ into the function and set this equal to 33,

$$f(1) = 10(1)^3 + 8(1)^2 + C = 33$$

Simplifying and solving for C, we get

$$\begin{aligned} 10 + 8 + C &= 33 \\ C &= 15 \end{aligned}$$

This means that the function with a derivative of $f'(x) = 30x^2 + 16x$ and that passes through $f(1) = 33$ is

$$f(x) = 10x^3 + 8x^2 + 15$$

Following this strategy, we can find any function from its derivative and some ordered pair on the function.

Example 10 Find a Function from its Derivative

The derivative of a function $g(x)$ is

$$g'(x) = 2.4x + 17$$

Find the function $g(x)$ that passes through $(10,\ 295.4)$.

Solution To find a function from its derivative, take the antiderivative:

$$\begin{aligned} g(x) &= \int (2.4x + 17)\, dx \\ &= 2.4 \cdot \frac{x^2}{2} + 17x + C \\ &= 1.2x^2 + 17x + C \end{aligned}$$

Since $(10,\ 295.4)$ is on $g(x)$, we know that $g(10) = 295.4$. Substitute these values into the function and solve for C. This gives

$$\begin{aligned} g(10) = 1.2(10)^2 + 17(10) + C &= 295.4 \\ 120 + 170 + C &= 295.4 \\ C &= 5.4 \end{aligned}$$

Using this value of C gives

$$g(x) = 1.2x^2 + 17x + 5.4$$

We can also apply this strategy to marginal functions in economics. From the marginal function, we can calculate the original function using the antiderivative.

Example 11 Find the Profit Function

A small manufacturing function models its marginal profit with the function

$$P'(Q) = -0.0070Q + 29.14 \quad \text{dollars per unit}$$

where Q is the number of units produced and sold. If producing and selling 2000 units results in \$34,280 in profit, find the profit function $P(Q)$ in dollars.

Solution The antiderivative of $P'(Q)$ yields $P(Q)$. Carry out the indefinite integral to give

$$\begin{aligned} P(Q) &= \int (-0.0070Q + 29.14)\, dQ \\ &= -0.0070 \cdot \frac{Q^2}{2} + 29.14Q + C \\ &= -0.0035Q^2 + 29.14Q + C \end{aligned}$$

We know that a quantity of 2000 units produced and sold leads to a profit of \$34,280 or $P(2000) = 34280$. Putting these values into the profit function gives us the appropriate value for C:

$$\begin{aligned} P(2000) = -0.0035(2000)^2 + 29.14(2000) + C &= 34280 \\ -14000 \quad + \quad 58280 \quad + C &= 34280 \\ C &= -10000 \end{aligned}$$

The corresponding profit function is

$$P(Q) = -0.0035Q^2 + 29.14Q - 10000$$