Question 2: How is the area under a function approximated?

In the previous question, we found the areas of rectangles of rectangles to estimate the change in a quantity based on its rate of change. Since the heights of the rectangles come from the data points, we can only use a limited number of rectangles. If the rate is given by a function (the derivative), we have as many data points for the rate as we want. In this case, we may increase the number of rectangles in the left and right hand sums.

Example 4 Find Area with Left and Right Hand Sums

The rate at which revenue changes at the electric car company is given by the derivative,

$$R'(Q) = 60.0200\,Q^{-0.0432} \text{ thousand dollars per electric car}$$

where Q is the number of electric cars produced.

a. Find the change in revenue from increasing production from 90 to 110 electric cars using left and right hand sums. Use four rectangles in the sums.

Solution In Question 1, we defined the sums with terms of the form $f(x_i)\Delta x$ .In this example, the independent variable is Q and the rate is $R'(Q)$. We need to form a sum where each term has the form $R'(Q_i)\Delta Q$.

A change in production from 90 to 110 is an increase of $110 - 90 = 20$ electric cars. Each rectangle must be $\frac{20}{4} = 5$ wide or $\Delta Q = 5$.

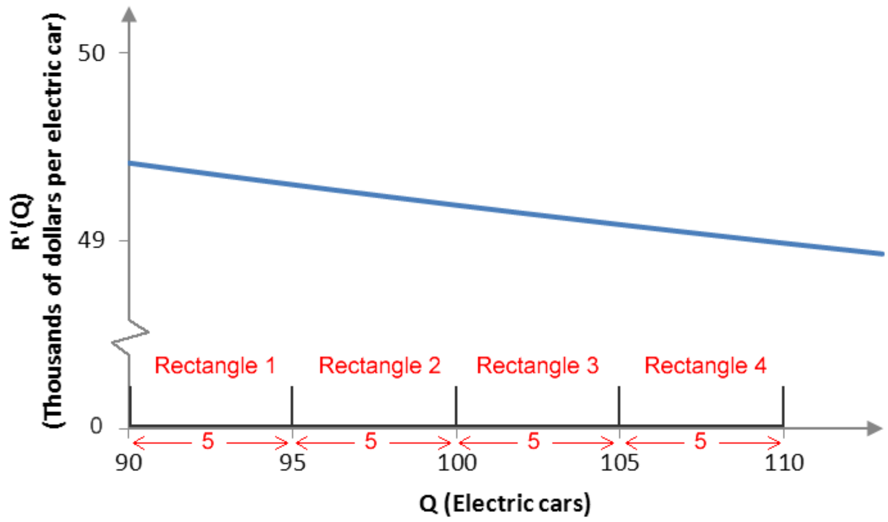

Figure 7 – The bases of the four rectangles in part a.

In Figure 7, the base of each rectangle is graphed. Depending on the type of sum, each rectangle will touch the rate function on the left of right hand side. In a left hand sum, the first rectangle will touch the rate function at $Q_1 = 90$, the second rectangle at $Q_2 = 95$, the third rectangle at $Q_3 = 100$, and the fourth rectangle at $Q_4 = 105$.

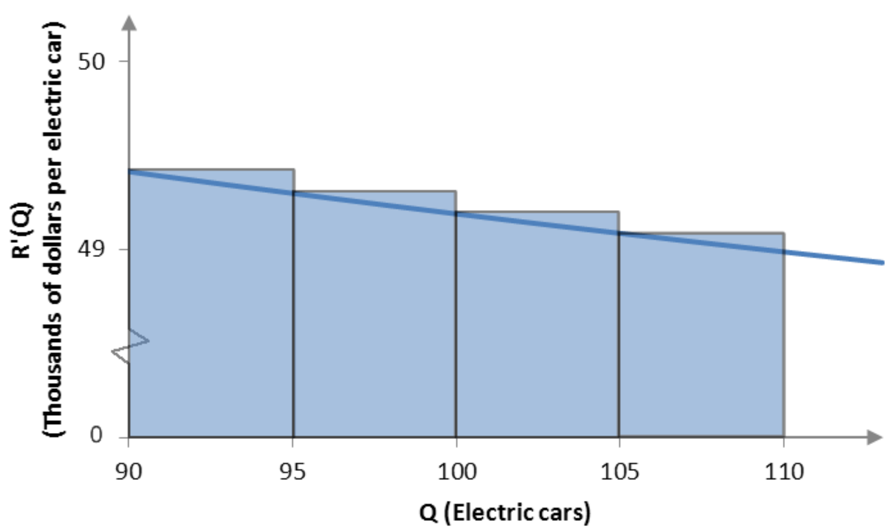

Figure 8 – The rectangles corresponding to a left hand sum.

In terms of the rate function $R'(Q)$, the left hand sum is

$$\text{Left Hand Sum} = R'(90)\Delta Q + R'(95)\Delta Q + R'(100)\Delta Q + R'(105)\Delta Q$$
$$= 49.417 \cdot 5 + 49.301 \cdot 5 + 49.192 \cdot 5 + 49.089 \cdot 5$$
$$\approx 984.993$$

The right hand sum is calculated using the rate on the right side of each rectangle.

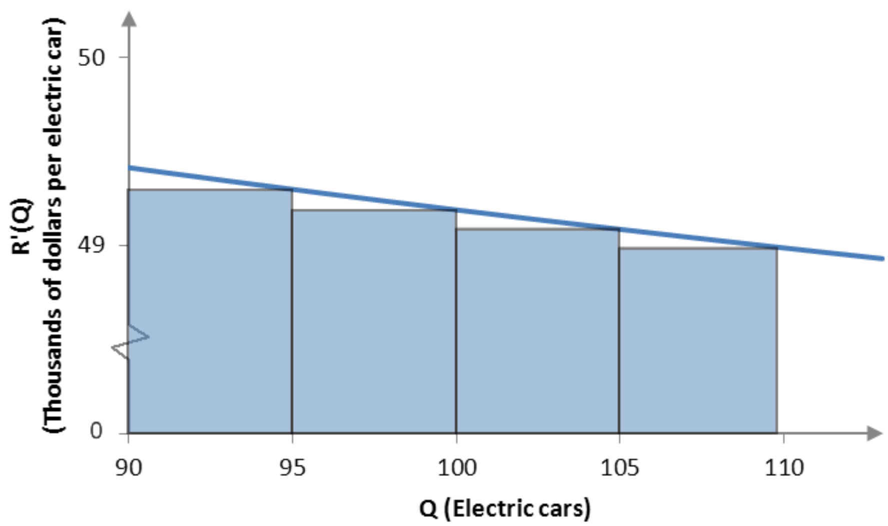

Figure 9 – The rectangles corresponding to a right hand sum.

The terms in the sum are

$$\text{Right Hand Sum} = R'(95)\Delta Q + R'(100)\Delta Q + R'(105)\Delta Q + R'(110)\Delta Q$$
$$= 49.301 \cdot 5 + 49.192 \cdot 5 + 49.089 \cdot 5 + 48.990 \cdot 5$$
$$\approx 982.861$$

Based on these sums, the change in revenue is between 982.861 and 984.993 thousand dollars.

b. Find the change in revenue from increasing production from 90 to 110 electric cars using left and right hand sums. Use eight rectangles in the sums.

Solution To create the sums with eight rectangles, we need to find the width of each rectangle. Since we need to form eight rectangles from $Q = 90$ to $Q = 110$, each rectangle must have a width of

$$\Delta Q = \frac{110 - 90}{8} = 2.5$$

In this expression, the numerator contains the difference in the production levels. The denominator contains the number of rectangles that will be placed between these levels.

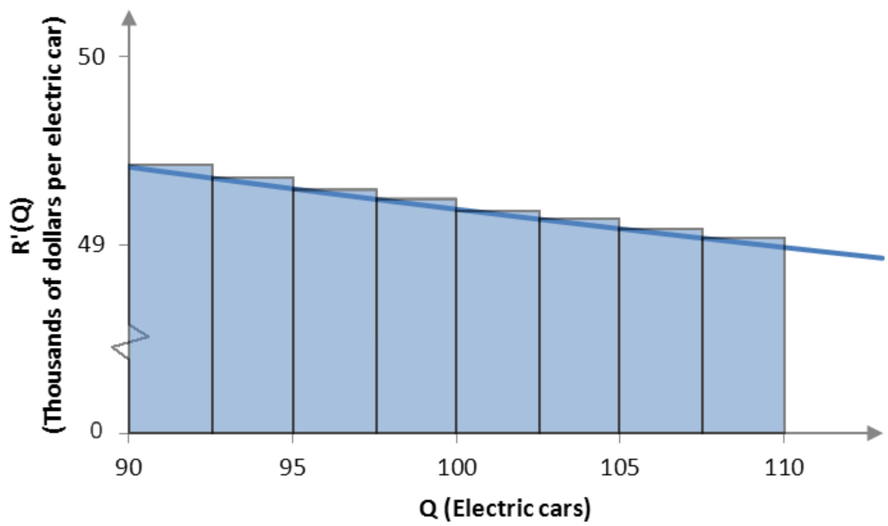

Figure 10 – The rectangles corresponding to a left hand sum with eight rectangles.

In the left hand sum, the rectangle height are determined by the rate at $Q = 90, 92.5, 95, \ldots, 107.5$. The left hand sum is

$$\text{Left Hand Sum} = R'(90)\Delta Q + R'(92.5)\Delta Q + R'(95)\Delta Q + \cdots + R'(107.5)\Delta Q$$

$$\approx 984.453$$

The heights of the rectangles in a right hand sum are determined by the rate at $Q = 92.5, 95, 97.5, \ldots, 110$.

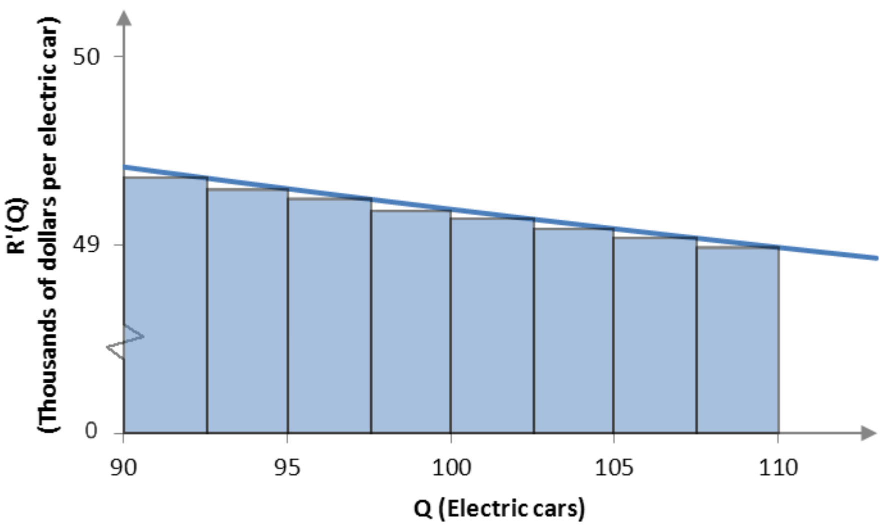

Figure 11 - The rectangles corresponding to a right hand sum with eight rectangles.

The right hand sum is

$$\text{Right Hand Sum} = R'(92.5)\Delta Q + R'(95)\Delta Q + R'(97.55)\Delta Q + \cdots + R'(110)\Delta Q$$

$$\approx 983.387$$