Question 2: How is the definite integral related to the approximate area?

In Section 13.2, we approximated the area under the derivative

$$R'(Q) = 60.0200\,Q^{-0.0432} \text{ thousand dollars per electric car}$$

to find the change in revenue when production is changed from 90 cars to 110 cars. Using left and right sums, we were able to find an upper and lower bound for this change. When the number of rectangles is increased, these bounds move closer and closer together. For a large enough number of rectangles, we can make the estimates match to any number of decimal places.

As the number of rectangles increases, it is hard to show every term in a left of right sum. Symbolically, we can use summation notation to indicate a sum. In summation notation, the Greek letter sigma is used to indicate a sum. After the sigma a template for the terms is written to establish the pattern in the sum. For instance, the expression

$$f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

can be written in summation notation as

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

Each term in the sum is a product of a function value $f(x_i)$ and Δx. The index i indicates what changes in each term of the sum. In this case, i changes from 1 to n so the terms contain different values of x from x_1 through x_n.

Example 3 Write with Summation Notation

In Example 4, we formed a sum

$$R'(Q_1)\Delta Q + R'(Q_2)\Delta Q + \cdots + R'(Q_8)\Delta Q$$

Write this sum with summation notation.

Solution This sum has eight terms. Each term is a product of the rate R' and ΔQ. This forms the basic pattern to the terms. Each term has a different Q value so we can write the sum as

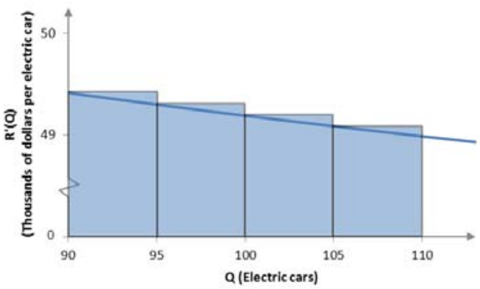
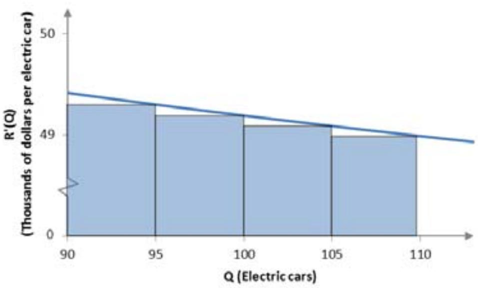
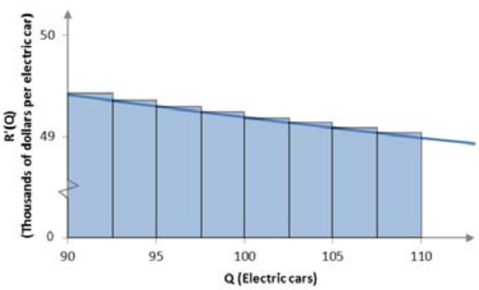
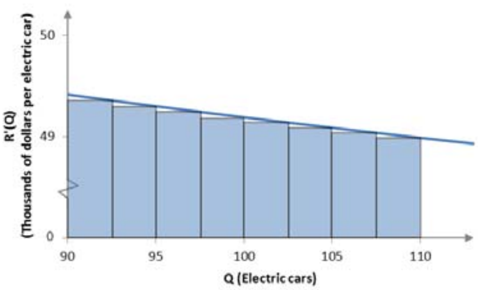
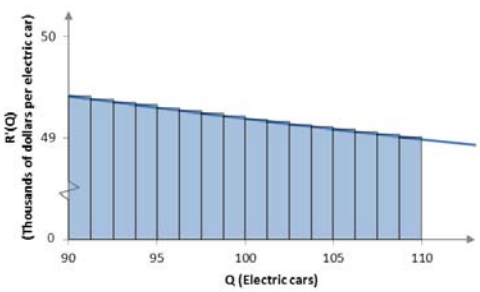
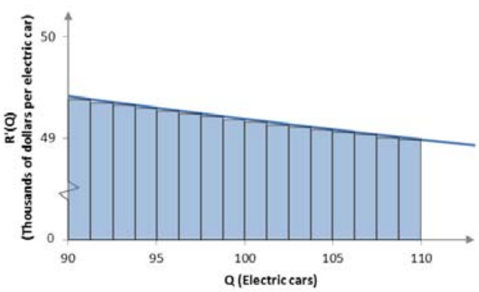
$$\sum_{i=1}^{8} R'(Q_i)\Delta Q$$

As the number of rectangles increases, the left and right hand estimates get closer together. In Example 2 and Example 4 of Section 13.2, we found the left and right hand sums for 2, 4 and 8 rectangles. The table below shows these estimates and several others for larger numbers of rectangles.

Number of Rectangles	Left Hand Sum	Right Hand Sum
2	986.090	981.820
4	984.993	982.861
8	984.453	983.387
16	984.185	983.652
32	984.051	983.784
64	983.984	983.851

As the number of rectangles is increased, the left and right hand sums get closer and closer together. Rounded to the nearest integer, the sums are exactly the same, 984. For larger and larger number of rectangles, the sums will match to more and more decimal places. If we were to fit 4500 rectangle from 90 to 110, both estimates would be the same to three decimal places, 983.918.

Let's look at some of these estimates graphically.

Number of Rectangles	Left Hand Sum	Right Hand Sum
4		
8		
16		

As the number of rectangles increases, the top of each rectangle looks more and more like the rate function. The area of the rectangles gets closer and closer to the area under the function and above the horizontal axis between 90 and 110.

If we were able to use an infinite number of rectangles, each with an infinitely small width, the area of these rectangles would be equal to the area under the function.

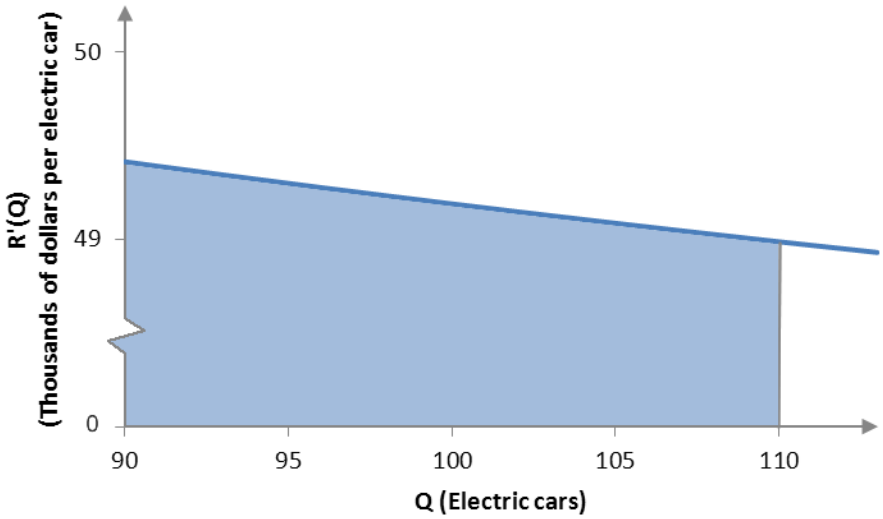


Figure 6 – As the number of rectangles increases, the area of the rectangles gets closer and closer to the area under the function, above the horizontal axis, from 90 to 110.

Although we can't sum up the infinite number of rectangles directly, we can symbolize this sum using a limit,

$$\text{Exact Area} = \lim_{n \to \infty} \sum_{i=1}^{n} R'\left(Q_i\right)\Delta Q$$

In the expression on the right, we are summing up the area of n rectangles whose heights are $R'\left(Q_i\right)$ and widths are ΔQ. By taking the limit as n approaches infinity, we are describing the values the sum of the areas approach as we use more and more rectangles. In practice, we don't need to actually compute this sum for an infinite number of rectangles. We simply use enough rectangles so that the sums match to a specified number of decimal places.

A definite integral is used to symbolize the exact area under a function over some interval.

Let $f(x)$ be a function defined on an interval $[a,b]$. The definite integral of f from a to b,

$$\int_a^b f(x)\,dx$$

denotes the exact area under the function $f(x)$, above the x axis, and between a and b. This area is defined by

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ and x_i is an x value in the i^{th} rectangle. This limit must exist for the area to be defined.

The function $f(x)$ is called the integrand and the symbol $\int$ is called an integral.

We can use the definite integral notation to locate the area on a graph. For the change in revenue from $Q=90$ to $Q=110$, we write

$$\int_{90}^{110} R'(Q)\,dQ = \lim_{n\to\infty}\sum_{i=1}^{n} R'(Q_i)\Delta Q$$

The definite integral on the left represents the exact area under $R'(Q)$ and above the horizontal axis over the interval $[90,110]$. The right hand sum represents the area of rectangles whose heights are determined by $R'(Q_i)$ and widths are ΔQ. For now, we'll use left, right, or midpoint sums to estimate the area of this region. However, in later sections we'll find the exact value of the area using antiderivatives.

Example 4 Estimate the Definite Integral

Estimate the definite integral $\int_0^5 (x+1)\,dx$ using left and right hand sums with 5 rectangles.

Solution To estimate the area under $f(x)=x+1$, we will calculate the Riemann sum

$$\sum_{i=1}^{5} f(x_i)\Delta x$$

The width of each rectangle is $\Delta x = \frac{5-0}{5} = 1$

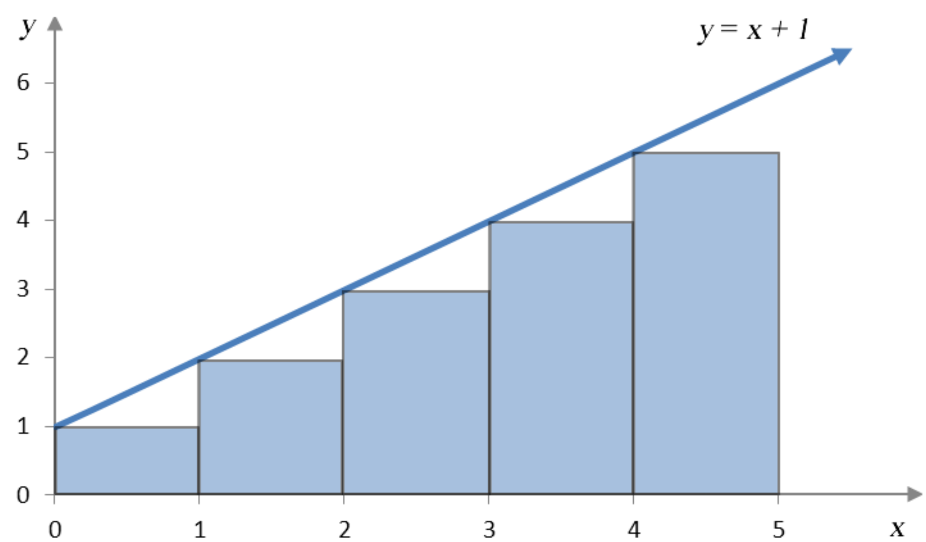


Figure 7 – The left hand sum for 5 rectangles on the interval [0,5].

From the graph in Figure 7, we see that $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, $x_4 = 3$, and $x_5 = 4$. This means that

$$\sum_{i=1}^{5} f(x_i)\Delta x = f(0)\Delta x + f(1)\Delta x + f(2)\Delta x + f(3)\Delta x + f(4)\Delta x$$

$$= 1\cdot 1 + 2\cdot 1 + 3\cdot 1 + 4\cdot 1 + 5\cdot 1$$

$$= 15$$

For the right hand sum, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$, and $x_5 = 5$.

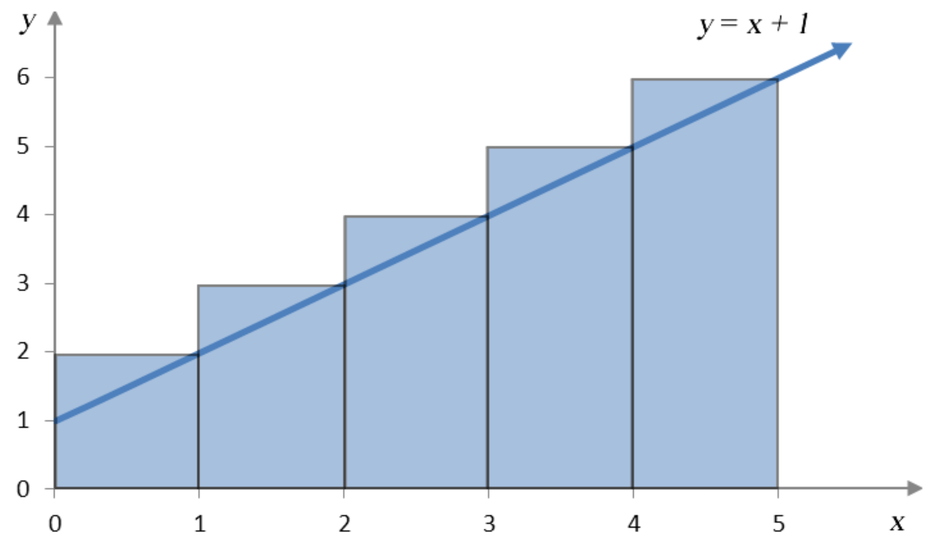


Figure 8 - The right hand sum for 5 rectangles on the interval [0,5].

In this case, the sum is

$$\sum_{i=1}^{5} f(x_i)\Delta x = f(1)\Delta x + f(2)\Delta x + f(3)\Delta x + f(4)\Delta x + f(5)\Delta x$$

$$= 2\cdot 1 + 3\cdot 1 + 4\cdot 1 + 5\cdot 1 + 6\cdot 1$$

$$= 20$$

These estimates place the exact area between 15 and 20. In Example 2, we found the exact area geometrically as 17.5. In general, we can't find the area geometrically so the left and right hand sums give us a convenient way of estimating the area. In section 13.4, we'll learn how to find the area exactly for integrands whose antiderivative we can compute.

Example 5 Estimate the Definite Integral

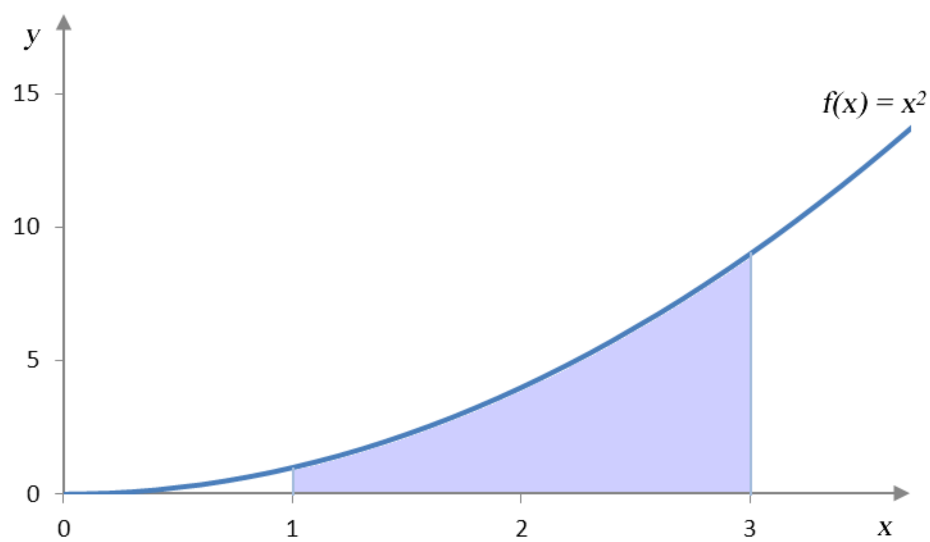
The definite integral

$$\int_1^3 x^2 dx$$

denotes the area of some region.

a. Graph the region corresponding to the definite integral.

Solution The integrand of this definite integral is $f(x) = x^2$. The region corresponding to the definite integral is the area between the function and the x-axis from $x = 1$ to $x = 3$.



b. Estimate the definite integral using a midpoint sum with four rectangles.

Solution Each rectangle must have a width of $\Delta x = \frac{3-1}{4} = 0.5$. The rectangles heights come from the rate at the midpoint of each rectangle. For instance, the first rectangle extends from $x = 1$ to $x = 1.5$ so the midpoint is $x = 1.25$. The area of the corresponding rectangle is $1.25^2 \cdot 0.5$.

The midpoint sum is

$$\sum_{i=1}^{4} f(x_i)\Delta x = 1.25^2 \cdot 0.5 + 1.75^2 \cdot 0.5 + 2.25^2 \cdot 0.5 + 2.75^2 \cdot 0.5$$

$$= 8.625$$

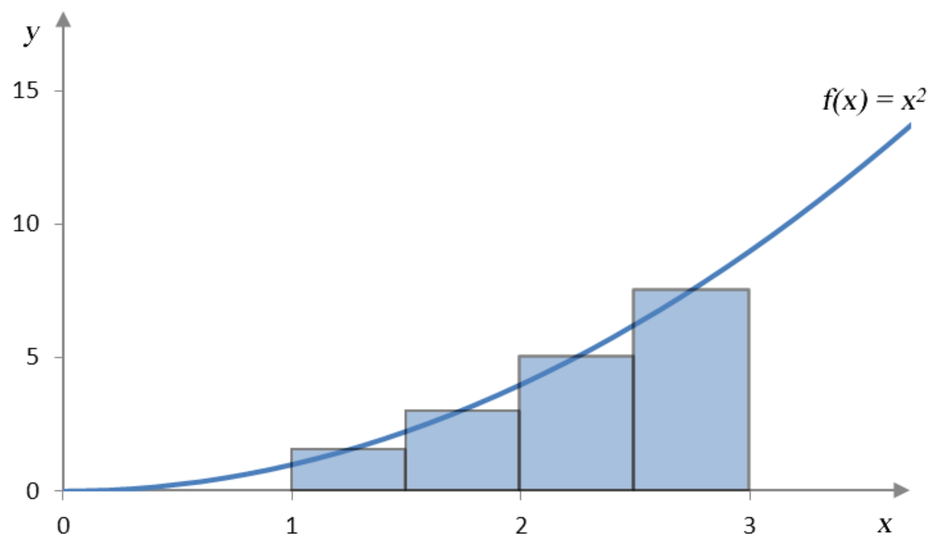


Figure 9 – The rectangles for the midpoint sum where the width of each rectangle is 0.5.