Question 1: How do we find the antiderivative of functions involving compositions?

To reverse the Chain Rule derivative $\frac{d}{dx}\left[\left(x^2+10\right)^7\right]=7(x^2+10)^6\cdot 2x$, we must find the antiderivative

$$\int 7\left(x^2+10\right)^6\cdot 2x\,dx$$

without knowledge of the derivative. We need to locate the composition in the integrand corresponding to $f'\left(g\left(x\right)\right)$. Examining the integrand, notice that the function x^2+10 is composed with the function $7x^6$. To apply the u-substitution method, let

$$u=x^2+10$$

The derivative of this function is

$$\frac{du}{dx}=2x$$

If we solve for the symbol du, we get

$$dx\cdot\frac{du}{dx}=2x\,dx \qquad\rightarrow\qquad du=2x\,dx$$

Now let's use the u and du in the integrand.

$$\int 7\underbrace{\left(x^2+10\right)}_{u}{}^6\cdot\underbrace{2x\,dx}_{du}=\int 7u^6du$$

The integrand is written in terms of the new variable u. It is much simpler in terms of u instead of x. Using the power rule for antiderivatives, we can carry out the antiderivative,

$$\begin{aligned}\int 7\left(x^2+10\right)^6\cdot 2x\,dx&=\int 7u^6du\\&=7\cdot\frac{1}{7}u^7+C\\&=u^7+C\end{aligned}$$

Once the antiderivative is complete, substitute the expression for u into the antiderivative to yield

$$\int 7\left(x^2+10\right)^6 \cdot 2x\, dx = \left(x^2+10\right)^7 + C$$

As demonstrated above, this integration technique looks fairly complex. However, we can put this all together in a format that is more reader friendly. This is more like what you would produce on paper.

$$\int 7\underbrace{\left(x^2+10\right)}_{u}^6 \cdot \underbrace{2x\, dx}_{du} = \int 7u^6\, du$$

$$= 7\int u^6\, du$$

$$= 7\cdot\frac{1}{7}u^7 + C$$

$$= u^7 + C$$

$$= \left(x^2+10\right)^7 + C$$

$$u = x^2+10 \quad \rightarrow \quad dx\cdot\frac{du}{dx} = 2x\, dx$$

$$du = 2x\, dx$$

Each of the steps we carried out is included in this short calculation. The u-substitution is emphasized by putting it in a box. You are encouraged to adopt a similar format.

Example 1 Find the Antiderivative

Evaluate $\int \left(x^3+x^2\right)^4 \left(3x^2+2x\right) dx$.

Solution The integrand contains a factor that is a composition of two functions. For this reason, choose

$$u = x^3 + x^2$$

With this substitution, we can find the corresponding derivative.

$$\frac{du}{dx} = 3x^2 + 2x$$

$$du = \left(3x^2 + 2x\right) dx$$

The integrand may be rewritten in terms of u as

$$\int \underbrace{\left(x^3 + x^2\right)}_{u}{}^4 \underbrace{\left(3x^2 + 2x\right) dx}_{du} = \int u^4 \, du$$

Now we can evaluate the antiderivative using the Power Rule for Antiderivatives.

$$\int \left(x^3 + x^2\right)^4 \left(3x^2 + 2x\right) dx = \int u^4 \, du$$

$$= \frac{1}{5} u^5 + C$$

Insert the expression for u to complete the antiderivative,

$$\int \left(x^3 + x^2\right)^4 \left(3x^2 + 2x\right) dx = \frac{1}{5} \left(x^3 + x^2\right)^5 + C$$

These steps are summarized below.

$$\int \underbrace{\left(x^3 + x^2\right)}_{u}{}^4 \underbrace{\left(3x^2 + 2x\right) dx}_{du} = \int u^4 \, du$$

$$= \frac{1}{5} u^5 + C$$

$$= \frac{1}{5} \left(x^3 + x^2\right) + C$$

$$u = x^3 + x^2 \quad \rightarrow \quad dx \cdot \frac{du}{dx} = \left(3x^2 + 2x\right) \cdot dx$$

$$du = \left(3x^2 + 2x\right) dx$$

Example 2 Find the Antiderivative

Evaluate $\int 16(2t+1)^3 \, dt$.

Solution The factor $(2t+1)^3$ is a composition of the functions $2t+1$ and t^3. The inside function for the composition is $2t+1$ so let

$$u = 2t+1$$

Even though the variable in this problem is t, this does not change the strategy for carrying out u-substitution. We only need to alter the derivative notation to account for the variable name. The derivative is

$$\frac{du}{dt} = 2$$

Solving for du, we get

$$du = 2\,dt$$

To substitute these expressions for u and du, we expect to see $2\,dt$ in the integrand instead of 16 and dt. To remedy, write 16 as the factors $8 \cdot 2$,

$$\int 16(2t+1)^3 \, dt = \int 8(2t+1)^3 \, 2\,dt$$

Notice that the 2 is written next to dt. It is easy to see how to substitute u and du in this rewritten form,

$$\int 8\underbrace{(2t+1)}_{u}{}^3 \, \underbrace{2\,dt}_{du} = \int 8u^3 du$$

The antiderivative of $8u^3$ is $8\cdot\frac{1}{4}u^4+C$. In terms of t, we get $2(2t+1)^4+C$. Let's summarize these steps:

$$\begin{aligned}\int 16(2t+1)^3\,dt &= \int 8(2t+1)^3\,2\,dt \\ &= 8\int u^3\,du \\ &= 8\cdot\frac{1}{4}u^4+C \\ &= 2(2t+1)^4+C\end{aligned}$$

$$u=2t+1 \quad\rightarrow\quad dt\cdot\frac{du}{dt}=2\cdot dt$$
$$du=2\,dt$$

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In the previous examples, we rewrote an existing factor to make the u-substitution easier to recognize. In the next example, we introduce new factors in the integrand to make the u-substitution easier to carry out.

Example 3 Find the Antiderivative

Evaluate $\int u\,e^{u^2}\,du$.

Solution The variable in this example is u. This can cause confusion since we want to use the letter u to name the substitution. However, there is nothing special about the letter u. We can use a different letter to name the substituion. In this case, let

$$w=u^2$$

Find dw by taking the derivative with respect to u:

$$\frac{dw}{du}=2u \quad\rightarrow\quad du\cdot\frac{dw}{du}=2u\cdot du$$
$$dw=2u\,du$$

We would like to see the factors $2u$ in the integrand. But all we see is u. To fix this situation, we introduce a factor of 2 along with a corresponding factor of $\frac{1}{2}$.

$$\int u\, e^{u^2}\, du = \int \tfrac{1}{2}\, e^{\overbrace{u^2}^{w}}\, \underbrace{2u\, du}_{dw}$$

Together these factor equal 1 so the integrand has not been changed. The 2 and u have been written next to the du to make the substitution easier to recognize.

$$\int \tfrac{1}{2} e^{u^2}\, 2u\, du = \int \tfrac{1}{2} e^{w}\, dw$$

The antiderivative is $\tfrac{1}{2} e^{w} + C$. After replacing w with u^2, we have $\tfrac{1}{2} e^{u^2} + C$. Let's summarize all of the steps described above:

$$\begin{aligned} \int u\, e^{u^2}\, du &= \int \tfrac{1}{2} u\, e^{u^2}\, 2\, du \\ &= \tfrac{1}{2} \int e^{w}\, dw \\ &= \tfrac{1}{2} e^{w} + C \\ &= \tfrac{1}{2} e^{u^2} + C \end{aligned}$$

$$w = u^2 \quad \rightarrow \quad du \cdot \frac{dw}{du} = 2u \cdot du$$
$$dw = 2u\, du$$

In this last example, the composition is not obvious. To find the substitution, we often look at the different parts of the integrand. This helps us to decide if one part of the integrand is related to the derivative of the other.

Example 4   Find the Antiderivative

Evaluate $\int \frac{t^4}{t^5+1}\,dt$.

Solution Either of the two pieces of the integrand might be the substitution u. Note that the derivative of the denominator t^5+1 is $5t^4$. This is five times the numerator so the du will take up this part of the integrand.

Let $u = t^5+1$ so that

$$\frac{du}{dt} = 5t^4$$

$$du = 5t^4\,dt$$

The du contains an extra factor of 5. We'll rewrite the integrand to include this factor,

$$\int \frac{t^4}{t^5+1}\,dt = \frac{1}{5}\int \frac{1}{t^5+1}\cdot 5t^4\,dt$$

The extra factor of $\frac{1}{5}$ is included to balance the extra 5 in the integrand. With this revision, we substitute u and du,

$$\frac{1}{5}\int \frac{1}{\underbrace{t^5+1}_{u}}\cdot \overbrace{5t^4\,dt}^{du} = \frac{1}{5}\int \frac{1}{u}\,du$$

The antiderivative of $\frac{1}{u}$ is $\ln|u|+C$. Including the factor of $\frac{1}{5}$ and the expression for u gives the solution to the original integral,

$$\int \frac{t^4}{t^5+1}\,dt = \tfrac{1}{5}\ln\left|t^5+1\right|+C$$

The process illustrated above is summarized below.

$$\int \frac{t^4}{t^5+1}\,dt = \frac{1}{5}\int \frac{1}{t^5+1}\cdot 5t^4\,dt$$
$$= \frac{1}{5}\int \frac{1}{u}\,du$$
$$= \frac{1}{5}\ln|u| + C$$
$$= \frac{1}{5}\ln\left|t^5+1\right| + C$$

$$u = t^5 + 1 \quad \rightarrow \quad dt \cdot \frac{du}{dt} = 5t^4 \cdot dt$$
$$du = 5t^4\,dt$$

In the first three examples, we examined possible compositions for the u-substitution. In Example 4, the composition is not obvious. In this case, we examine different pieces of the integrand to see if the derivatives appear elsewhere in the integrand.