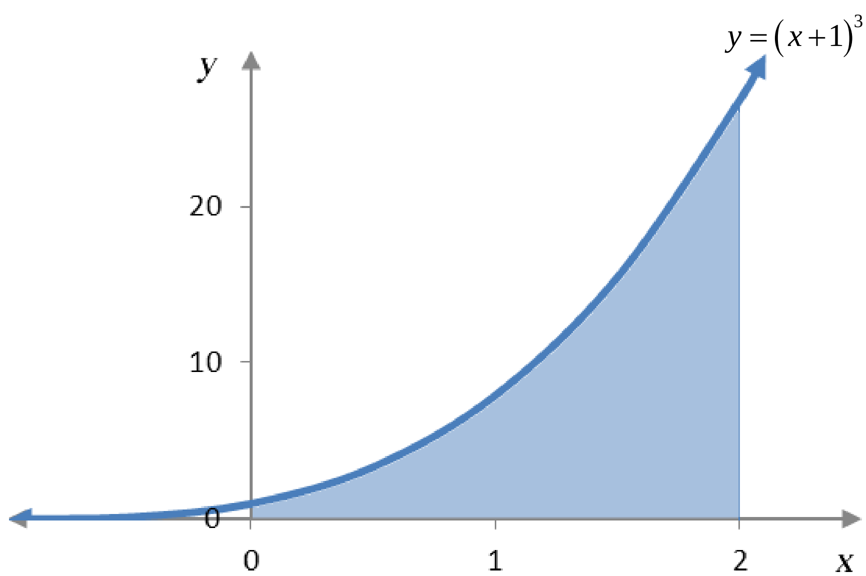
Question 2: How is the exact area under a function involving compositions computed?

The exact area under a function may be computed using the Fundamental Theorem of Calculus,

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is an antiderivative of $f(x)$. To apply this theorem, we need to find the antiderivative $F(x)$ of the integrand $f(x)$. For integrands involving compositions, we often use u-substitution to find the antiderivative.

Suppose we wish to find the area of the region below.



The area in this region is denoted by

$$\int_0^2 (x+1)^3\,dx$$

In the Fundamental Theorem of Calculus, the integrand plays the role of $f(x)$. The area is found by evaluating the antiderivative difference, $F(2) - F(0)$. Let's find the antiderivative using u-substitution:

$$\int (x+1)^3\,dx = \int u^3\,du$$
$$= \tfrac{1}{4}u^4 + C$$
$$= \tfrac{1}{4}(x+1)^4 + C$$

$$u = x+1 \quad \rightarrow \quad dx \cdot \frac{du}{dx} = 1 \cdot dx$$
$$du = dx$$

Using this antiderivative, the exact area is

$$\int_0^2 (x+1)^3\,dx = \tfrac{1}{4}(x+1)^4\Big|_0^2$$
$$= \tfrac{1}{4}(3)^4 - \tfrac{1}{4}(1)^4$$
$$= 20$$

In this calculation we use u-substitution to find the antiderivative in terms of x. Then we evaluate the antiderivative at $x = 0$ and $x = 2$ to find the area.

It is also possible to apply the Fundamental Theorem of Calculus as we carry out u-substitution. This is done by realizing that the substitution $u = x+1$ allows us to convert x values to u values.

$$\begin{matrix} x = 0 \\ x = 2 \end{matrix} \xrightarrow{u=x+1} \begin{matrix} u = 1 \\ u = 3 \end{matrix}$$

The definite integral in terms of x can also be written in terms of u,

$$\int_0^2 (x+1)^3\,dx = \int_1^3 u^3\,du$$

Instead of applying the Fundamental Theorem of Calculus to the left side, we can apply it to the right side:

$$\int_1^3 u^3\,du = \tfrac{1}{4}u^4\Big|_1^3$$
$$= \tfrac{1}{4}\cdot 3^4 - \tfrac{1}{4}\cdot 1^4$$
$$= 20$$

You may choose either strategy to calculate the definite integral. Evaluating the antiderivative in terms of x or u at the appropriate values leads to the same area. Some students prefer to work in terms of u since it often reduces the total number of steps in the calculation.

Example 5 Find the Area

Find the area under $y = e^{5x+4}$ and above the x axis between $x = 0$ and $x = 2$.

Solution This area corresponds to the definite integral

$$\int_0^2 e^{5x+4}\, dx$$

We start by using u-substitution to write the definite integral in terms of u. Start with the substitution

$$u = 5x + 4 \quad \rightarrow \quad dx \cdot \frac{du}{dx} = 5 \cdot dx$$

$$du = 5\, dx$$

The limits of integration for the substitution are found by putting each x value into $u = 5x + 4$:

$$\begin{array}{ccc} x = 0 & \xrightarrow{u=5x+4} & u = 4 \\ x = 2 & & u = 14 \end{array}$$

Now rewrite the integrand and limits to evaluate the definite integral.

$$\int_0^2 e^{5x+4}\,dx = \tfrac{1}{5}\int_0^2 e^{5x+4}\cdot 5\,dx$$ Insert extra factors of $\frac{1}{5}$ and 5

$$= \tfrac{1}{5}\int_4^{14} e^u\,du$$ Change $5x+4$ to u and $5\,dx$ to du

$$= \tfrac{1}{5}e^u\Big|_4^{14}$$ The antiderivative of e^u is e^u

$$= \tfrac{1}{5}e^{14} - \tfrac{1}{5}e^4$$

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Example 6 Evaluate the Definite Integral

Evaluate the definite integral $\int_0^1 \frac{6x^2}{x^3+1}\,dx$.

Solution There are two potential choices for the u-substitution. Let's try $u = 6x^2$. In this case,

$$u = 6x^2 \quad \rightarrow \quad dx \cdot \frac{du}{dx} = 12x \cdot dx$$

$$du = 12x\,dx$$

The rest of the integrand contains terms with x^3 so this substitution does not look useful.

The second choice for the u-substitution is $u = x^3 + 1$. For this substitution,

$$u = x^3 + 1 \quad \rightarrow \quad dx \cdot \frac{du}{dx} = 3x^2 \cdot dx$$

$$du = 3x^2\,dx$$

Since the numerator of the integrand is twice du, this looks like a good choice for the u-substitution.

Let's also change the limits of integration on the definite integral:

$$\begin{matrix} x=0 \\ x=1 \end{matrix} \xrightarrow{u=x^3+1} \begin{matrix} u=1 \\ u=2 \end{matrix}$$

Use this information to rewrite the definite integral and carry out the solution,

$$\int_0^1 \frac{6x^2}{x^3+1}\,dx = 2\int_0^1 \frac{1}{x^3+1}\cdot 3x^2\,dx \qquad \text{Rewrite } 6x^2 \text{ as } 2\cdot 3x^2$$

$$= 2\int_1^2 \frac{1}{u}\,du \qquad \text{Change } x^3+1 \text{ to } u \text{ and } 3x^2 \text{ to } du$$

$$= 2\ln(u)\Big|_1^2 \qquad \text{The antiderivative of } \tfrac{1}{u} \text{ is } \ln(u)$$

$$= 2\ln(2) - 2\ln(1)$$

$$= 2\ln(2)$$