Question 2: How is the exact area under a function involving products computed?

When we find the area under a function, we evaluate a definite integral. This may be done by approximating the area with left, right, or midpoint sums. If the integrand can be written as a product, we may be able to use integration by parts to find the antiderivative. Then we use the Fundamental Theorem of Calculus and evaluate the antiderivative at the limits of integration.

Example 4 Find the Area

Find the area of the region bounded by $y = \frac{\ln(x)}{x^2}$, the x axis, and $x = 5$.

Solution Start by graphing the region.

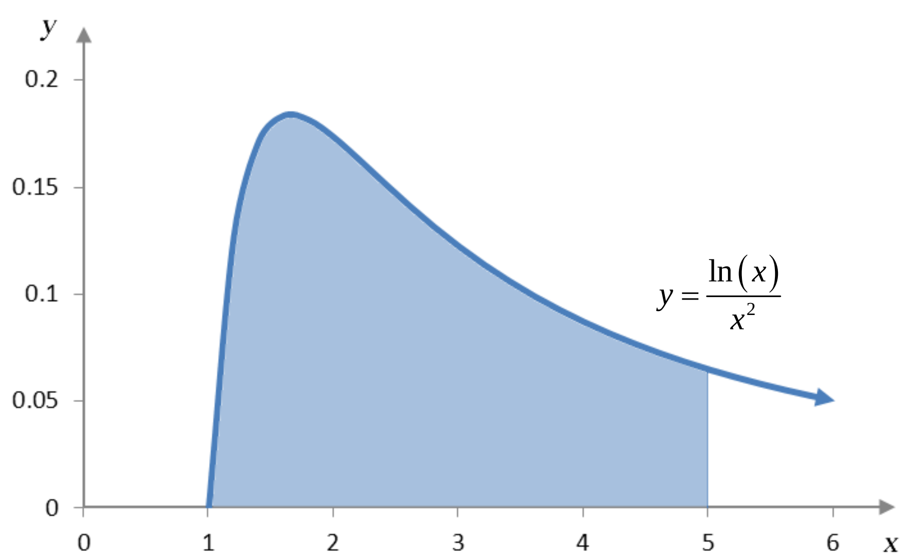

Figure 1 - The region described in Example 4.

This area corresponds to the definite integral

$$\int_1^5 \frac{\ln(x)}{x^2}\, dx$$

To apply the Fundamental Theorem of Calculus, we must find the antiderivative,

$$\int \frac{\ln(x)}{x^2}\,dx = \ln(x)\cdot\frac{-1}{x} - \int \frac{1}{x}\cdot\frac{-1}{x}\,dx$$
$$= -\frac{\ln(x)}{x} + \int x^{-2}\,dx$$
$$= -\frac{\ln(x)}{x} - \frac{1}{x} + C$$

$$u = \ln(x) \xrightarrow{\text{Derivative}} u' = \frac{1}{x}$$
$$v' = \frac{1}{x^2} = x^{-2} \xrightarrow{\text{Antiderivative}} v = \frac{-1}{x}$$

Use this antiderivative and the limits of integration to evaluate the definite integral,

$$\int_1^5 \frac{\ln(x)}{x^2}\,dx = \left(-\frac{\ln(x)}{x} - \frac{1}{x}\right)\Bigg|_1^5$$
$$= \left(-\frac{\ln(5)}{5} - \frac{1}{5}\right) - \left(-\frac{\ln(1)}{1} - \frac{1}{1}\right)$$
$$= -\frac{\ln(5)}{5} + \frac{4}{5}$$
$$\approx 0.4781$$

The area of the enclosed region is approximately 0.4781. ■

Definite integral are also used to compute the change in a function from the function's derivative. This is equivalent to finding the area under the derivative.

Example 5 Find the Change in Cost

When a small startup company produces x auxiliary powers units, its marginal cost is

$$C'(x) = -0.2640\ln(x) + 5.1790 \text{ tens of thousands of dollar per unit}$$

If production is increased from 20 to 25 units, how will the cost change?

Solution To answer this question, we must apply the Fundamental Theorem of Calculus to the marginal cost,

$$\int_{20}^{25} C'(x)\,dx = C(25) - C(20)$$

The right side describes the change in production. To calculate the definite integral, we first calculate the antiderivative using integration by parts. Although it is not obvious what the parts of the product are, try $u = -0.2640\ln(x) + 5.1790$ and $v' = 1$. This gives us

$$\begin{aligned} u &= -0.2640\ln(x) + 5.179 \xrightarrow{\text{Derivative}} u' = \frac{-0.2640}{x} \\ v' &= 1 \xrightarrow{\text{Antiderivative}} v = x \end{aligned}$$

With these factors, apply the integration by parts rule,

$$\begin{aligned} \int (-0.2640\ln(x) + 5.1790)\,dx &= (-0.2640\ln(x) + 5.1790)x - \int \tfrac{-0.2640}{x} \cdot x\,dx \\ &= (-0.2640\ln(x) + 5.1790)x - \int -0.2640\,dx \\ &= (-0.2640\ln(x) + 5.1790)x + 0.2640x + C \\ &= -0.2640x\ln(x) + 5.1790x + 0.2640x + C \\ &= -0.2640x\ln(x) + 5.4430x + C \end{aligned}$$

Substitute the limits of integration and subtract,

$$\begin{aligned} \int_{20}^{25} (-0.264\ln(x) + 5.179)\,dx &= (-0.264x\ln(x) + 5.443x)\Big|_{20}^{25} \\ &\approx 114.8304 - 93.0425 \\ &\approx 21.7879 \end{aligned}$$

The heights on the marginal cost function are in tens of thousands of dollars per unit. The widths are in units. This means that the area has units of

$$\frac{\text{tens of thousands of dollars}}{\text{unit}} \cdot \text{unit} = \text{tens of thousands of dollars}$$

A value of 21.7879 for the definite integral tells us that the cost increases by $217,879 when production is increased from 20 units to 25 units. ■