

Section 10.1 Functions

Question 1 – What is a function?

Question 2 – What is function notation?

Question 3 - What is a piecewise defined function?

Question 1 – What is a function?

Key Terms

Function

Linear function

Independent variable (input)

Dependent variable (output)

Summary

Many business functions can be written as linear functions. Linear functions have the form $y = mx + b$ where x is the independent variable that represents the input. The dependent variable is y and represents the output. The graph of a linear function is a line whose slope is m and vertical intercept is b .

By being a function, we know that each value of the independent variable corresponds to one value of the dependent variable. Simply put, in a function each input matches up with one output. There is nothing special about the names of the variables. Typically, we use names that reflect what the variable represents like P for price and Q for quantity in supply and demand functions. In the case of a demand function with Q as the independent variable, we would write the linear function (as a function of Q) as $P = mQ + b$.

Notes

Guided Example

Practice

Does the equation $y^2 - x = 5$ correspond to a function of x ?

Solution To decide if this equation corresponds to a function of x , we need to understand what is meant by “function of x ”. Since it says, “of x ”, we know that the independent variable or input is the variable x .

For this to correspond to a function of x , each value for the input x must correspond to exactly one value of the output y . This is easier to check if we solve the equation for the output.

To solve for y , add x to both sides of the equation:

$$y^2 = x + 5$$

Now square root both side of the equation. Doing this should yield a positive root and a negative root,

$$\sqrt{y^2} = \pm\sqrt{x+5}$$

$$y = \pm\sqrt{x+5}$$

The \pm means that any reasonable input we make for x will results in two values for y . For instance, $x = 4$ will give $y = \pm 3$. This might surprise you at first, but let’s check each of these in the original equation:

$$3^2 - 4 \stackrel{?}{=} 5$$

$$(-3)^2 - 4 \stackrel{?}{=} 5$$

Both of these statements are true. So, the input $x = 4$ corresponds to $y = 3$ and $y = -3$. In other words, there is an input that corresponds to more than one output. Because of this, the equation does not correspond to a function.

1. Does the equation $x^2 + \sqrt{y} = 9$ correspond to a function of x ?

2. Does the equation $\frac{1}{y^2} = x$ correspond to a function of x ?

Guided ExamplePractice

- a. Does the demand equation $4P + 5Q = 200$ correspond to a linear function of Q ?

Solution If a demand equation can be written in the form $P = mQ + b$, it is a linear function of Q . To see if this is possible, solve the equation for P . Start by subtracting $5Q$ from both sides:

$$4P = -5Q + 200$$

Now divide each term by 4 to get the P by itself on the left side.

$$\frac{4P}{4} = \frac{-5Q}{4} + \frac{200}{4}$$

$$P = -\frac{5}{4}Q + 50$$

Since the equation may be written in the proper format with $m = -\frac{5}{4}$ and $b = 50$, the equation does correspond to a linear function of Q .

- b. Does the demand function $4P + 5Q = 200$ correspond to a linear function of P ?

Solution This problem is very similar to the part above, except that it specifies a linear function of P . For this equation to correspond to a linear function of P , we must be able to rewrite it in the form $Q = mP + b$. Let's check to see if this is possible by solving the equation for Q . Start by subtracting $4P$ from both sides of the equation:

$$5Q = -4P + 200$$

Now divide each term by 5:

$$\frac{5Q}{5} = \frac{-4P}{5} + \frac{200}{5}$$

$$Q = -\frac{4}{5}P + 40$$

Since the equation may be written in the proper format with $m = -\frac{4}{5}$ and $b = 40$, the demand equation does correspond to a linear function of P .

3. Check to see whether the following are linear functions.

- a. Does the demand equation $20P + 4Q = 68$ correspond to a linear function of Q ?

- b. Does the demand equation $20P + 4Q = 68$ correspond to a linear function of P ?

Question 2 – What is function notation?

Key Terms

Function notation	Revenue Function
Demand function	Cost Function
Supply function	Profit Function
Equilibrium point	Quadratic function

Summary

Business and finance applications are filled with many different functions representing different quantities. To help us distinguish the function from each other, we give them names in place of the dependent variable. For instance, instead of writing $y = x^2$ we would write something like

$$f(x) = x^2$$

where f is the name of the function and x is the name of the independent variable. This might seem confusing at first, but it will be very handy when we are working with problems in which there are three functions. In this case, we can give each function a different name.

To indicate that we want to substitute a value into a function, we replace the independent variable with a value. For instance, $f(5)$ means substitute $x = 5$ into the function to give $f(5) = 5^2$ or 25. This tells you that the input of $x = 5$ corresponds to an output of $y = 25$.

In applications using supply and demand functions, we use $S(Q)$ to denote the supply function and $D(Q)$ to denote the demand function. Since the input is represented by the quantity Q , the output must correspond to the price. The equilibrium point is where the supply function is equal to the demand function or $S(Q) = D(Q)$.

Revenue, cost and profit are named with $R(Q)$, $C(Q)$, and $P(Q)$. Since the difference between revenue and cost is profit, we can write

$$P(Q) = R(Q) - C(Q)$$

To find a revenue function, multiply the demand $D(Q)$ by the quantity Q or $R(Q) = Q \cdot D(Q)$.

Break-even points are where the revenue and cost are equal. In other words, the profit is zero.

A quadratic function is any equation that can be written in the form $y = ax^2 + bx + c$ where a , b , and c are constants, x is the independent variable and y is the dependent variable. The graph of a quadratic function is a parabola whose vertex is located at $x = -\frac{b}{2a}$.

Notes

Guided ExamplePractice

Suppose that $D(Q)$ and $S(Q)$ are the demand and supply functions,

$$D(Q) = 9 - 0.75Q$$

$$S(Q) = 0.75Q$$

where Q is the quantity of some product produced.

- a. Find the price at which consumers will demand 4 or 8 products.

Solution The question gives two quantities and asks for the corresponding price from the demand function. Since the demand function takes in the quantity Q and gives out the price, we simply put in the values.

$$D(4) = 9 - 0.75(4) = 6$$

$$D(8) = 9 - 0.75(8) = 3$$

At a price of \$6, consumers demand a quantity of 4. At a price of \$3, consumers demand a quantity of 8. This makes sense since we would expect the consumer to demand more product when the price is low.

- b. Find the price at which suppliers will supply 4 or 8 products.

Solution As with the previous part, we need to substitute 4 or 8 into the function. But since we are asking about suppliers, we put the values into the supply function.

$$S(4) = 0.75(4) = 3$$

$$S(8) = 0.75(8) = 6$$

At a price of \$3, suppliers supply a quantity of 4. At a price of \$6, suppliers supply a quantity of 8. This makes sense since suppliers want to create more product when the price is high.

4. Suppose that $D(Q)$ and $S(Q)$ are the demand and supply functions,

$$D(Q) = 16 - 1.25Q$$

$$S(Q) = 0.75Q$$

where Q is the quantity of some product produced.

- a. Find the price at which consumers will demand 5 or 10 products.

- b. Find the price at which suppliers will supply 5 or 10 products.

- c. At a price of \$6, what is the quantity demanded by consumers and the quantity that suppliers are willing to produce?

Solution In this part we are given a price (output) and we need to know the corresponding quantity. For consumers, at a price of \$6:

$$6 = 9 - 0.75Q$$

$$-3 = -0.75Q$$

$$4 = Q$$

For suppliers, we set the supply function equal to 6 and solve for Q:

$$6 = 0.75Q$$

$$8 = Q$$

At a price of \$6, consumers demand a quantity of 4 and suppliers want to produce a quantity of 8. This market has a surplus at \$6.

- d. What is the equilibrium price?

Solution The equilibrium point is where the supply and demand functions are equal. Setting $S(Q) = D(Q)$ allows us to solve for Q :

$$0.75Q = 9 - 0.75Q$$

$$1.5Q = 9$$

$$Q = 6$$

You might be anxious to say this is the answer. But this is the equilibrium quantity, not equilibrium price. To get the corresponding price, we need to put $Q = 6$ into the supply or demand function:

$$D(6) = 9 - 0.75(6) = 4.5$$

$$S(6) = 0.75(6) = 4.5$$

It doesn't matter which function we use since they should both give us an equilibrium price of \$4.5.

- c. At a price of \$10, what is the quantity demanded by consumers and the quantity that supplier are willing to produce?

- d. What is the equilibrium price?

Guided ExamplePractice

The demand function for Q units of a product is given by

$$D(Q) = 9 - 0.75Q$$

The cost function is given by the function

$$C(Q) = 1.75Q + 10$$

- a. Find the revenue function $R(Q)$.

Solution Multiplying the demand function (which outputs price) times the quantity Q gives the revenue function.

$$\begin{aligned} R(Q) &= Q \cdot D(Q) \\ &= Q(9 - 0.75Q) \\ &= 9Q - 0.75Q^2 \end{aligned}$$

Note that this is a quadratic function with $a = -0.75$, $b = 9$ and $c = 0$.

- b. Find the break-even point(s)?

Solution The break-even point is where the revenue is equal to the cost, $R(Q) = C(Q)$. Setting the functions equal, we get

$$9Q - 0.75Q^2 = 1.75Q + 10$$

To solve this equation, we need to move all the terms to one side:

$$-0.75Q^2 + 7.25Q - 10 = 0$$

This is a quadratic equation with $a = -0.75$, $b = 7.25$, and $c = -10$. Putting these values into the quadratic equation gives

$$Q = \frac{-7.25 \pm \sqrt{7.25^2 - 4(-0.75)(-10)}}{2(-0.75)}$$

$$\approx 1.67, 8$$

5. The demand function for Q units of a product is given by

$$D(Q) = 16 - 1.25Q$$

The cost function is given by the function

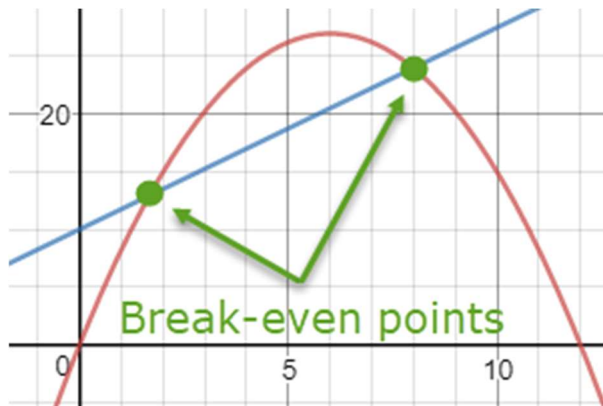
$$C(Q) = 2Q + 15$$

- a. Find the revenue function $R(Q)$.

- b. Find the break-even point(s)?

- c. On a graph of $C(Q)$ and $R(Q)$, where do the break-even points lie?

Solution



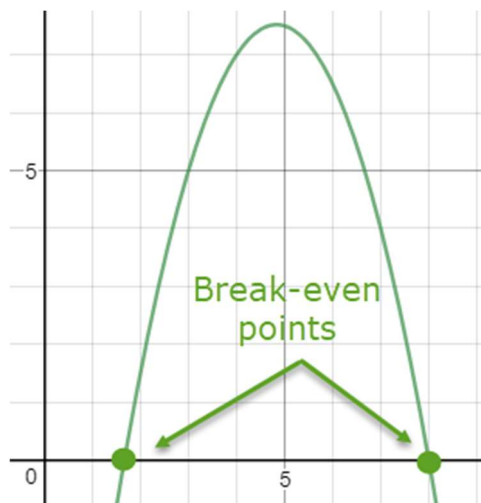
- d. Find the profit function $P(Q)$.

Solution Profit is the difference between revenue and cost, $P(Q) = R(Q) - C(Q)$. Being careful to distribute the subtraction, we get

$$\begin{aligned} P(Q) &= (9Q - 0.75Q^2) - (1.75Q + 10) \\ &= -0.75Q^2 + 7.25Q - 10 \end{aligned}$$

- e. Where do the break-even points lie on the graph of $P(Q)$?

Solution



- c. On a graph of $C(Q)$ and $R(Q)$, where do the break-even points lie?

- d. Find the profit function $P(Q)$.

- e. Where do the break-even points lie on the graph of $P(Q)$?

Question 3 – What is a piecewise defined function?

Key Terms

Piecewise defined function

Summary

A piecewise function is a function that is built up in sections. Each section has a formula with which you compute the output values. In addition, there is a description to the right of each formula that tells you when to use the formula.

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 10 \\ 0.5x + 5 & \text{if } 10 \leq x < 20 \\ 0.25x + 10 & \text{if } x \geq 20 \end{cases}$$

Formulas for each piece

Where to apply the
formula for each piece

Notes

Guided Example

A credit union offers 2% annual interest on the first \$10,000 in an account and 4% annual interest on the additional money over \$10,000. The annual interest earned on D dollars is

$$I(D) = \begin{cases} 0.02D & \text{for } 0 \leq D \leq 10,000 \\ 0.02(10,000) + 0.04(D - 10,000) & \text{for } D > 10,000 \end{cases}$$

- a. If the account contains \$9500, what is the annual interest?

Solution This part gives a dollar amount of $D = 9500$. This amount lies in the top section of the function. Putting the value into the formula gives $I(9500) = 0.02(9500) = 190$.

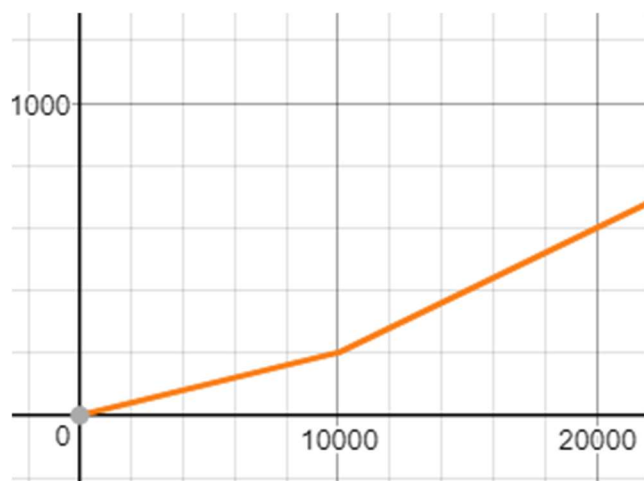
- b. If the account contains \$11,000, what is the annual interest?

Solution The amount $D = 11,000$ lies in the bottom section of the formula. Put this value into the formula to give $I(11,000) = 0.02(10,000) + 0.04(11,000 - 10,000) = 240$.

- c. Graph the function.

Solution To graph this function, graph several points in each section.

D	$I(D)$
0	0
5000	100
9500	190
11000	240
15000	400
20000	600



Practice

6. A credit union offers 1% annual interest on the first \$5,000 in an account and 2% annual interest on the additional money over \$5,000. The annual interest earned on D dollars is

$$I(D) = \begin{cases} 0.01D & \text{for } 0 \leq D \leq 5,000 \\ 0.01(5,000) + 0.02(D - 5,000) & \text{for } D > 5,000 \end{cases}$$

- a. If the account contains \$4000, what is the annual interest?

- b. If the account contains \$7,000, what is the annual interest?

- c. Graph the function.

Guided Example

In 2008, the following description was used to calculate state income taxes in Alabama.

For the first \$500 in taxable income, the tax rate on that income is 2%.

For the next \$2500 in taxable income, the tax rate on that income is 4%.

On all additional income, the tax rate on that income is 5%.

Use this information to find the information below.

- a. Find the tax on \$400 in taxable income.

Solution To find the tax, we need to multiply the amount of taxable income by the rate for that income. On \$400 of taxable income,

$$\text{Tax} = 0.02(400) = 8$$

- b. Find the tax on \$900 of taxable income.

Solution We need to consider two tax rates. The first \$500 of income is taxed at a rate of 2%. The next \$400 of taxable income is taxed at a rate of 4%. This means the tax on \$900 of taxable income is

$$\text{Tax} = 0.02(500) + 0.04(400) = 26$$

- c. Find the tax on \$4000 of taxable income.

Solution The tax on \$4000 of taxable income comes from three tax rate: \$500 at a 2% tax rate, \$2500 at a 4% tax rate, and \$1000 at a 5% tax rate. So the tax on \$4000 of taxable income is

$$\text{Tax} = 0.02(500) + 0.04(2500) + 0.05(1000) = 160$$

- d. Model the tax as a function of taxable income x with a piecewise linear function.

Solution Since there are three different tax rates, we'll use a piecewise linear function with three pieces in it. The three rates apply to certain ranges of taxable income, so start by writing the function as

$$T(x) = \begin{cases} ?? & \text{for } 0 \leq x \leq 500 \\ ?? & \text{for } 500 < x \leq 3000 \\ ?? & \text{for } x > 3000 \end{cases}$$

This reflects the information on where the different tax rates apply.

To find the formula for each piece, it is easiest to simply find two point in each piece and calculate the equation of a line through those pieces. For instance, we already know that the point $(400, 8)$ is on the first section of the graph from part a. We also know that if a person has no income, they must pay no tax so $(0, 0)$ is also on this section. The slope between these points is

$$\text{slope} = \frac{8-0}{400-0} = 0.02$$

Since the graph passes through $(0,0)$, the vertical intercept is 0. Putting this information into the function gives

$$T(x) = \begin{cases} 0.02x & \text{for } 0 \leq x \leq 500 \\ ?? & \text{for } 500 < x \leq 3000 \\ ?? & \text{for } x > 3000 \end{cases}$$

Now let's look at the next section. In part b we found that \$900 in taxable income yields \$26 of tax so $(900, 26)$ is on the second section of the function. Further, we can also determine that taxable income of \$3000 leads to a tax of

$$\text{Tax} = 0.02(500) + 0.04(2500) = 110$$

so $(3000, 110)$ is also on the second section. The slope between these points is

$$\text{slope} = \frac{110-26}{3000-900} = 0.04$$

If we think of this section as having the equation $y = 0.04x + b$, we can find the value of b by substituting one of the ordered pairs into the function. This gives

$$110 = 0.04(3000) + b$$

$$110 = 120 + b$$

$$-10 = b$$

This yields the equation $y = 0.04x - 10$.

You can avoid using two points if you utilize the word description carefully. We know that the first \$500 has a 2% tax and income above \$500 (but below \$3000) has a 4% tax. This means that for any taxable income x between \$500 and \$3000 has a tax of

$$\begin{aligned} \text{Tax} &= \underbrace{0.02(500)}_{\text{tax on first \$500}} + \underbrace{0.04(x-500)}_{\text{tax on income above \$500}} \\ &= 10 + 0.04x - 20 \\ &= 0.04x - 10 \end{aligned}$$

Using these values in the tax function gives

$$T(x) = \begin{cases} 0.02x & \text{for } 0 \leq x \leq 500 \\ 0.04x - 10 & \text{for } 500 < x \leq 3000 \\ ?? & \text{for } x > 3000 \end{cases}$$

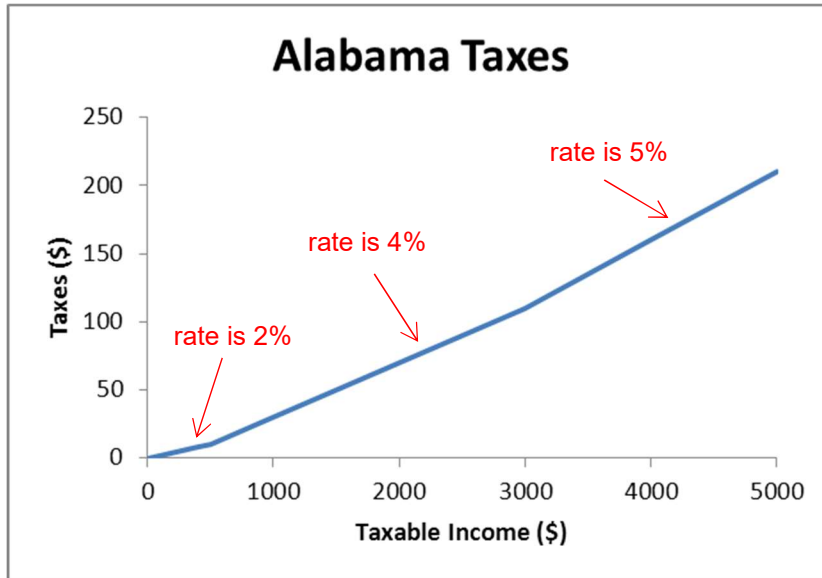
Let's use the same strategy on the last section. We already know that $(4000, 160)$ is on the third section of the function from part c. A little math could also demonstrate that $(5000, 210)$ is also on the graph. The slope between these points is

$$\text{slope} = \frac{210-160}{5000-4000} = 0.05$$

The vertical intercept is found by solving $210 = 0.05(5000) + b$ to give $b = -40$. With this information, we write the tax function as

$$T(x) = \begin{cases} 0.02x & \text{for } 0 \leq x \leq 500 \\ 0.04x - 10 & \text{for } 500 < x \leq 3000 \\ 0.05x - 40 & \text{for } x > 3000 \end{cases}$$

The graph of this function is shown below.



Notice that as the rate goes up, the graph gets steeper.

Practice

7. The following description was proposed to calculate state income tax.

For the first \$1000 in taxable income, the tax rate on that income is 1%.

For the next \$5000 in taxable income, the tax rate on that income is 3%.

On all additional income, the tax rate on that income is 5%.

Use this information to find the information below.

a. Find the tax on \$800 in taxable income.

b. Find the tax on \$3000 in taxable income.

c. Find the tax on \$8000 in taxable income.

d. Model the tax as a function of taxable income x with a piecewise linear function.

Section 10.2 Functions

Question 1 – How do you evaluate a limit from a table?

Question 2 – How do you evaluate a limit from a graph?

Question 1 – How do you evaluate a limit from a table?

Key Terms

Left-hand limit

One-sided limit

Right-hand limit

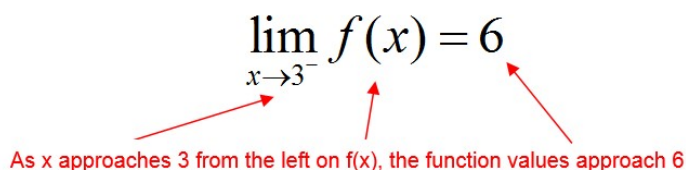
Two-sided limit

Summary

Limits are used to describe the behavior of a function near a point. In particular, they help us to decide the output from a function as inputs get very close to some value from the left or right. This is how a limit looks.

$$\lim_{x \rightarrow 3^-} f(x) = 6$$

As x approaches 3 from the left on f(x), the function values approach 6



By examining the output from the function near the designated input, we can look for patterns. You may locate these patterns from a table, a graph, or using algebra. Since a “-” follows the 3 in the limit, this is a left-hand limit and inputs get closer and closer to 3 from below. If the number had been followed by a “+”, this would be a right-hand limit. In this case we would look at x values that are slightly larger than 3 but getting closer and closer.

If the one-sided limits at a point are equal, then the two-sided limit is equal to the same numbers. For instance, if

$$\begin{array}{ll} \text{Left hand limit} & \lim_{x \rightarrow 3^-} f(x) = 6 \\ \text{Right hand limit} & \lim_{x \rightarrow 3^+} f(x) = 6 \end{array} \Rightarrow \lim_{x \rightarrow 3} f(x) = 6 \quad \text{Two-sided limit}$$

However, if the left-hand limit and the right-hand limit are not equal, then the two-sided limit does not exist (DNE).

If the output does not get closer to a value as the x values get closer and closer, we also say that the limit does not exist. If the outputs do this by growing larger and larger, we say the limit approaches ∞ . If the outputs grow more and more negative, we say that the limit approaches $-\infty$.

Notes

Guided Example

If $g(x) = \frac{x^2 - 4}{x - 2}$, evaluate $\lim_{x \rightarrow 2} g(x)$

Solution This limit asks us to look at what happens to outputs from $\frac{x^2 - 4}{x - 2}$ as x values get closer and closer to 2. To help us see what happens, make a table of values on either side of $x = 2$.

x approaches 2 from the left

x approaches 2 from the right

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$g(x)$	3.9	3.99	3.999	undefined	4.001	4.01	4.1

output gets closer and closer to 4

output gets closer and closer to 4

Looking at x values on the left side of 2, it appears that the outputs are increasing and getting closer and closer to 4. Based on this evidence, we can write the left-hand limit as

$$\lim_{x \rightarrow 2^-} g(x) = 4$$

Now let's check the right side of 2. In this part of the table, the outputs are decreasing but getting closer and closer to 4. Based on this evidence, we write the right-hand limit as

$$\lim_{x \rightarrow 2^+} g(x) = 4$$

Since both one-sided limits are equal, the two-sided limit is also equal to 4 and

$$\lim_{x \rightarrow 2} g(x) = 4$$

Practice

a. If $g(x) = \frac{\sqrt{x} - 2}{x - 4}$, evaluate $\lim_{x \rightarrow 4} g(x)$

Guided Example

If $h(x) = \frac{x}{x-1}$, evaluate $\lim_{x \rightarrow 1} h(x)$

Solution As in the previous guided example, we need to make a table of values on either side of $x = 1$. These values should get closer and closer to 1 from below and above.

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$h(x)$	-9	-99	-999	undefined	1001	101	11

If we examine the x values on the left side of $x = 1$, we see that the outputs from the function get more and more negative. Although there is a pattern, the numbers are not getting closer and closer to some value. Because of this, the left-hand limit as x approaches 1 does not exist. However, to indicate that the outputs are getting more and more negative, we would write

$$\lim_{x \rightarrow 1^-} h(x) = -\infty$$

Now let's look at x values on the right-hand side of $x = 1$. As we get closer and closer to 1, the outputs from the function grow larger and larger. Based on this evidence, we would say the limit does not exist. Furthermore, the right-hand limit is growing so we would also indicate

$$\lim_{x \rightarrow 1^+} h(x) = \infty$$

Since the one-sided limits do not match, the two-sided limit does not exist.

Practice

b. If $h(x) = \frac{\sqrt{x}-5}{x-3}$, evaluate $\lim_{x \rightarrow 3} h(x)$

Guided Example

A credit union offers 2% annual interest on the first \$10,000 in an account and 4% annual interest on the additional money over \$10,000. The annual interest earned on D dollars is

$$I(D) = \begin{cases} 0.02D & \text{for } 0 \leq D \leq 10,000 \\ 0.04D - 200 & \text{for } D > 10,000 \end{cases}$$

- a. Find the limit $\lim_{D \rightarrow 10000^-} I(D)$.

Solution Make a table of values on the left-hand side of $D = 10,000$.

D	9999	9999.9	9999.99	9999.999	10000
$I(D)$	199.98	199.998	199.9998	199.99998	200

As D gets closer and closer to \$10,000, the interest gets closer and closer to \$200. Based on this evidence, we would say

$$\lim_{D \rightarrow 10000^-} I(D) = 200$$

- b. Find the limit $\lim_{D \rightarrow 10000^+} I(D)$.

Solution Make a table of values on the right-hand side of $D = 10,000$.

D	10000	10000.001	10000.01	10000.1	10001
$I(D)$	200	200.00004	200.0004	200.004	200.04

As D gets closer and closer to \$10,000, the interest gets closer and closer to \$200. Based on this evidence, we would say

$$\lim_{D \rightarrow 10000^+} I(D) = 200$$

- c. Find the limit $\lim_{D \rightarrow 10000} I(D)$.

Solution Since the left-hand limit and the right-hand limit at $D = 10,000$ are equal, the two-sided limit is also equal to the same value,

$$\lim_{D \rightarrow 10000} I(D) = 200$$

Practice

- c. A credit union offers 1% annual interest on the first \$5,000 in an account and 2% annual interest on the additional money over \$5,000. The annual interest earned on D dollars is

$$I(D) = \begin{cases} 0.01D & \text{for } 0 \leq D \leq 5,000 \\ 0.02D - 50 & \text{for } D > 5,000 \end{cases}$$

- a. Find the limit $\lim_{D \rightarrow 5000^-} I(D)$.

- b. Find the limit $\lim_{D \rightarrow 5000^+} I(D)$.

- c. Find the limit $\lim_{D \rightarrow 5000} I(D)$.

Question 2 – How do you evaluate a limit from a graph?

Key Terms

Left-hand limit

Right-hand limit

One-sided limit

Two-sided limit

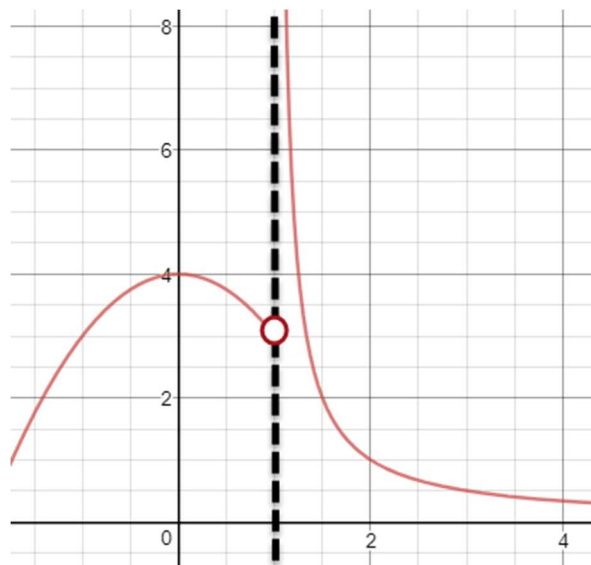
Summary

In the previous question, you examined the outputs of a function near a point using a table. You looked at the y values in the table on the left or right side of the specified x value, to determine the value of the limit. In this question, you do the exact same thing, except now you use the graph to determine the value of the left and right hand limits.

Notes

Guided Example

Find the limit $\lim_{x \rightarrow 1} f(x)$ using the graph of $f(x)$ below.



Solution To determine the value of the two-sided limit, we first need to look at the value of the one-sided limits. Let's start with the left-hand limit, $\lim_{x \rightarrow 1^-} f(x)$. In the graph, there is an asymptote at $x = 1$ and the function is not defined at that point. However, if you look slightly to the left of $x = 1$, say at about $x = 0.9$, you see the y value at that region is right around $y = 3$. As you get even closer to $x = 1$, the y values get closer and closer to 3. Therefore, $\lim_{x \rightarrow 1^-} f(x) = 3$.

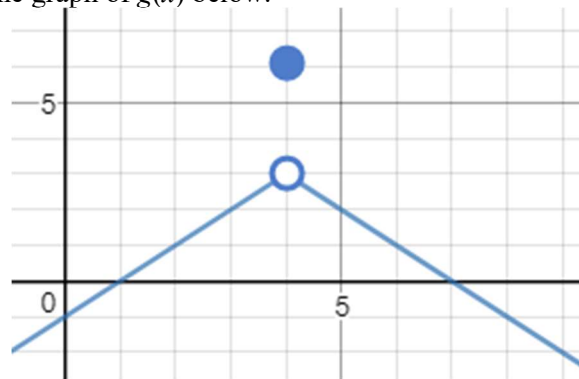
Now look at the other side of $x = 1$. Notice that the graph shoots upward on the right side of the asymptote. This means that the closer we get to $x = 1$, the larger the y values get. This means the limit does not exist. In fact, since the values get larger and larger, $\lim_{x \rightarrow 1^+} f(x) = \infty$.

Since the left and right hand limits do not match,

$$\lim_{x \rightarrow 1} f(x) = \text{does not exist}$$

Practice

1. Find the limits using the graph of $g(x)$ below.



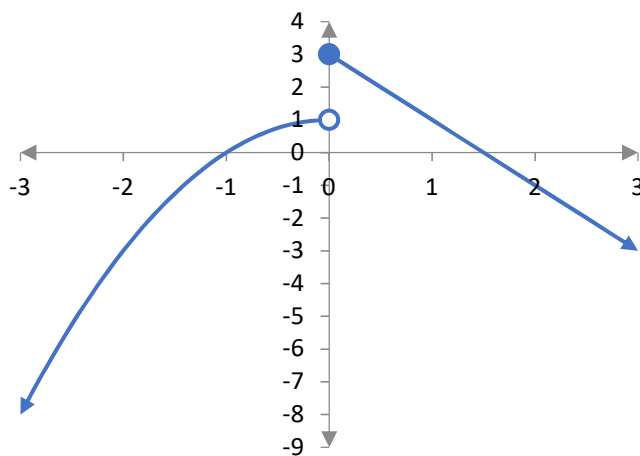
a. $\lim_{x \rightarrow 4^-} g(x)$

b. $\lim_{x \rightarrow 4^+} g(x)$

c. $\lim_{x \rightarrow 4} g(x)$

Practice

2. Find the limits using the graph of $g(x)$ below.



a. $\lim_{x \rightarrow 0^-} g(x)$

b. $\lim_{x \rightarrow 0^+} g(x)$

c. $\lim_{x \rightarrow 0} g(x)$

Section 10.3 Computing Limits Algebraically

Question 1 How can a limit be computed algebraically?

Question 2 How do you evaluate limits involving difference quotients?

Question 1 – How can a limit be computed algebraically?

Key Terms

Algebraically

Rationalize

Common denominator

Conjugate

Summary

When attempting to compute a limit, a good starting strategy is to simply substitute the x value into the expression. If you get a number, then you are done. That is the value of the limit.

Often you will get a 0 in the denominator and a nonzero number in the numerator. When this happens, you need to resort to one-sided limits and use a graph or table to estimate the value of the limit.

Suppose you substitute the x value into the function and obtain a zero in the numerator and denominator. In this case, you should carry out some algebra to see if the expression can be reduced. This may mean factoring, getting a common denominator, or rationalizing part of the fraction. Once the expression is simplified, you can substitute the x value to obtain the value of the limit.

Notes

Guided Example

Practice

Compute the value of the limit

$$\lim_{x \rightarrow 3} \frac{x^2 - 5x + 6}{x - 3}$$

Solution If you substitute $x = 3$ into the numerator and denominator of the fraction, both are equal to zero. This indicates that there must be some type of algebra that we can do to simplify the fraction. Let's try factoring:

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{x^2 - 5x + 6}{x - 3} &= \lim_{x \rightarrow 3} \frac{(x - 2) \cancel{(x - 3)}}{\cancel{x - 3}} \\ &= \lim_{x \rightarrow 3} x - 2 \end{aligned}$$

The factor that reduces is the factor that is causing the top and bottom to be zero. Without those factors we can substitute $x = 3$ into $x - 2$ to give

$$\lim_{x \rightarrow 3} \frac{x^2 - 5x + 6}{x - 3} = 1$$

1. Compute the value of the limit

$$\lim_{x \rightarrow 4} \frac{x^2 - 2x - 8}{x - 4}$$

Guided Example

Practice

Compute the value of the limit

$$\lim_{x \rightarrow 0} \frac{\frac{1}{x-6} + \frac{1}{6}}{x}$$

Solution When we substitute $x = 0$ into the expression, we get $\frac{0}{0}$ so there must be some algebra we can try to simplify the expression. In this case, we will get a common denominator of $6(x - 6)$ and simplify:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\frac{1}{x-6} + \frac{1}{6}}{x} &= \lim_{x \rightarrow 0} \frac{\frac{6}{6(x-6)} + \frac{x-6}{6(x-6)}}{x} \\ &= \lim_{x \rightarrow 0} \frac{\frac{6+x-6}{6(x-6)}}{x} && \text{Combine fractions} \\ &= \lim_{x \rightarrow 0} \frac{\frac{x}{6(x-6)}}{x} && \text{Add like terms} \\ &= \lim_{x \rightarrow 0} \frac{\cancel{x}}{6(x-6)} \cdot \frac{1}{\cancel{x}} && \text{Invert and multiply} \\ &= \lim_{x \rightarrow 0} \frac{1}{6(x-6)} \end{aligned}$$

Now that we have reduced the fraction, we can substitute $x = 0$ into the reduced expression to yield

$$\lim_{x \rightarrow 0} \frac{\frac{1}{x-6} + \frac{1}{6}}{x} = -\frac{1}{36}$$

2. Compute the value of the limit

$$\lim_{x \rightarrow 0} \frac{\frac{1}{4} - \frac{1}{x+4}}{x}$$

Guided Example

Practice

Compute the value of the limit

$$\lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{x - 1}$$

Solution As with earlier examples, substituting $x = 1$ leads to $\frac{0}{0}$. To simplify this fraction, multiply the top and bottom by the conjugate of the top, $\sqrt{x} + 1$:

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{x - 1} &= \lim_{x \rightarrow 1} \frac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{(x - 1)(\sqrt{x} + 1)} \\ &= \lim_{x \rightarrow 1} \frac{x - 1\sqrt{x} + 1\sqrt{x} - 1}{(x - 1)(\sqrt{x} + 1)} && \text{Foil the top} \\ &= \lim_{x \rightarrow 1} \frac{\cancel{x} - 1}{(\cancel{x} - 1)(\sqrt{x} + 1)} && \text{Combine like terms} \\ &= \lim_{x \rightarrow 1} \frac{1}{\sqrt{x} + 1}\end{aligned}$$

Now that the expression has been reduced, substitute $x = 1$ into the reduced expression to give

$$\lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{x - 1} = \frac{1}{2}$$

3. Compute the value of the limit

$$\lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4}$$

Question 2 – How do you evaluate limits involving difference quotients?

Key Terms

Difference quotient

Summary

Difference quotients have the form $\frac{f(a+h) - f(a)}{h}$. When they are used in limits, they typically lead to $\frac{0}{0}$. To evaluate a limit with a difference quotient, we need to simplify as in the previous question and substitute the value into the simplified expression.

Notes

Guided Example

Practice

Compute $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ where

$$f(x) = 4x + 3 \text{ and } a = 1.$$

Solution Start by evaluating $f(a)$ and $f(a+h)$ for the given function:

$$\begin{aligned} f(1) &= 4(1) + 3 \\ &= 7 \\ f(1+h) &= 4(1+h) + 3 \\ &= 7 + 4h \end{aligned}$$

Put these values into the difference quotient and simplify:

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \rightarrow 0} \frac{7 + 4h - 7}{h} \\ &= \lim_{h \rightarrow 0} \frac{4\cancel{h}}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} 4 \end{aligned}$$

Since there is no place to substitute $h = 0$ into the reduced difference quotient,

$$\lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = 4$$

1. Compute $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ where

$$f(x) = 8x - 1 \text{ and } a = -2.$$

Guided Example

Practice

Compute $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ where $f(x) = x^2 - 4$ and $a = 3$.

Solution Start by evaluating $f(a)$ and $f(a+h)$ for the given function:

$$\begin{aligned} f(3) &= 3^2 - 4 \\ &= 5 \\ f(3+h) &= (3+h)^2 - 4 \\ &= 9 + 6h + h^2 - 4 \\ &= 5 + 6h + h^2 \end{aligned}$$

Put these values into the difference quotient and simplify:

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \rightarrow 0} \frac{5 + 6h + h^2 - 5}{h} \\ &= \lim_{h \rightarrow 0} \frac{6h + h^2}{h} && \text{Combine like terms} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}(6+h)}{\cancel{h}} && \text{Factor and reduce} \\ &= \lim_{h \rightarrow 0} 6 + h \end{aligned}$$

Substitute $h = 0$ into the reduced difference quotient,

$$\lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} = 6$$

2. Compute $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ where $f(x) = x^2 - 1$ and $a = 2$.

Guided Example

Practice

Compute $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ where $f(x) = x^2 - x$

Solution Start by evaluating $f(x+h)$ for the given function:

$$\begin{aligned} f(x+h) &= (x+h)^2 - (x+h) \\ &= x^2 + 2xh + h^2 - x - h \end{aligned}$$

Put this expression into the difference quotient with

$f(x) = x^2 - x$ and simplify:

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x - h - x^2 + x}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2 - h}{h} && \text{Combine like terms} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}(2x + h - 1)}{\cancel{h}} && \text{Factor and reduce} \\ &= \lim_{h \rightarrow 0} 2x + h - 1 \end{aligned}$$

Substitute $h = 0$ into the reduced difference quotient,

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = 2x - 1$$

The limit comes out as a function of x since we don't have specific values like earlier examples.

3. Compute $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

where $f(x) = x^2 + 2x$.

Guided Example

Compute $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ where $f(x) = \frac{1}{x}$.

Solution Start by evaluating $f(x+h)$ for the given function:

$$f(x+h) = \frac{1}{x+h}$$

Put this expression into the difference quotient with

$f(x) = \frac{1}{x}$ and simplify. To simplify you will need a

common denominator of $x(x+h)$

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{-h}{x(x+h)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-\cancel{h}}{x(x+h)} \cdot \frac{1}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} \frac{-1}{x(x+h)}\end{aligned}$$

Substitute $h = 0$ into the reduced difference quotient,

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \frac{-1}{x^2}$$

Practice

4. Compute $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ where

$$f(x) = \frac{1}{2x}.$$

Section 10.4 Computing Limits at Infinity

Question 1 – How do you evaluate a limit at infinity using a table or graph?

Question 2 – How do you evaluate limits at infinity using algebra?

Question 1 – How do you evaluate a limit at infinity using a table or graph?

Key Terms

Limit at infinity

Summary

Like limits at a specific value, limits at infinity can be estimated using a table or a graph. Instead of using numbers close to a specific value, we use numbers that are positive and very large for a limit that approaches infinity.

x values are positive and get larger and larger as we move right in the table

x	100	1000	10,000	100,000
expression				

For limits that approach negative infinity, we use numbers that are negative and very large.

x values are negative and get larger and larger as we move right in the table

x	-100	-1000	-10,000	-100,000
expression				

By examining the value of the expressions in the table, we can determine if the expression get closer and closer to a value or grows.

The same process may be carried out on a graph is carried out by moving to the far right on a graph (x approaches positive infinity) or moving to the far left on a graph (x approaches negative infinity). By examining the corresponding expression values, you can determine if they are getting closer and closer to a value or growing.

Notes

Guided Example

Practice

Find the value of $\lim_{x \rightarrow \infty} \frac{6x^2 + 7}{3x^2 + x}$ using a table.

Solution Since the limits specifies that x approaches infinity, construct a table of larger and larger positive values.

x	100	1000	10000
$\frac{6x^2 + 7}{3x^2 + x}$	1.9936	1.9993	1.9999

As x grows larger, the value of the expression appears to get closer and closer to 2. Based on this information,

$$\lim_{x \rightarrow \infty} \frac{6x^2 + 7}{3x^2 + x} \approx 2$$

This is an estimate. To find the value exactly, we need to compute the limit algebraically.

1. Find the value of $\lim_{x \rightarrow \infty} \frac{5x^2 + x}{3x^2 + 2x}$ using a table.

Guided Example

Practice

Find the value of $\lim_{x \rightarrow -\infty} \frac{2x + 7}{3x^2 + x}$ using a table.

Solution Since the limits specifies that x approaches negative infinity, construct a table of larger and larger negative values.

x	-100	-1000	-10000
$\frac{2x + 7}{3x^2 + x}$	-0.0065	-0.0007	-0.00007

As x grows more negative, the value of the expression appears to get closer and closer to 0,

$$\lim_{x \rightarrow -\infty} \frac{2x + 7}{3x^2 + x} \approx 0$$

This is an estimate based on the table. To find the exact value of the limit, we need to compute the value algebraically.

2. Find the value of $\lim_{x \rightarrow \infty} \frac{2x + x}{3x^2 + 5x}$ using a table.

Find the value of $\lim_{x \rightarrow \infty} \frac{3x^2 + x}{2x + 7}$ using a table.

Solution Since the limits specifies that x approaches infinity, construct a table of larger and larger negative values.

x	100	1000	10000
$\frac{3x^2 + x}{2x + 7}$	145.41	1495.3	14995

As x grows more positive, the value of the expression appears to get larger and larger. It is not approaching any number so the limit does not exist. Since it does this by growing larger and larger, we write

$$\lim_{x \rightarrow \infty} \frac{3x^2 + x}{2x + 7} = \infty$$

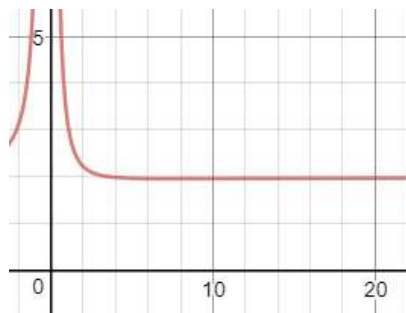
3. Find the value of $\lim_{x \rightarrow \infty} \frac{x^4 + 1}{3x + 6}$ using a table.

Notes

Guided Example

Find the value of the limit $\lim_{x \rightarrow \infty} \frac{6x^2 + 7}{3x^2 + x}$ using a graph.

Solution Start by making a graph of the expression for positive values.



As we move to the right along the graph, it appears that the graph is leveling off at about 2. This means as x gets larger and larger, the outputs are getting closer and closer to 2. This is a guess since we cannot distinguish between 2, 1.99 or 2.1. Based on this evidence, we write

$$\lim_{x \rightarrow \infty} \frac{6x^2 + 7}{3x^2 + x} \approx 2$$

To find the limit exactly, we need to find the limit algebraically.

Practice

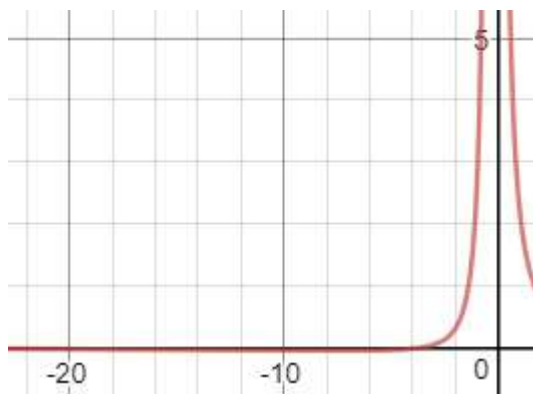
4. Find the value of $\lim_{x \rightarrow \infty} \frac{5x^2 + x}{3x^2 + 2x}$ using a graph.

Notes

Guided Example

Find the value of $\lim_{x \rightarrow -\infty} \frac{2x+7}{3x^2+x}$ using a graph.

Solution Start by making a graph of the expression for negative values.



Following the graph farther and farther to the left, it appears that the graph is leveling off at about 0. This means as x gets more and more negative, the outputs are getting closer and closer to 0. This is a guess since we cannot distinguish between 0, 0.1, -0.2. Based on this evidence, we write

$$\lim_{x \rightarrow -\infty} \frac{2x+7}{3x^2+x} \approx 0$$

To find the limit exactly, we need to find the limit algebraically.

Practice

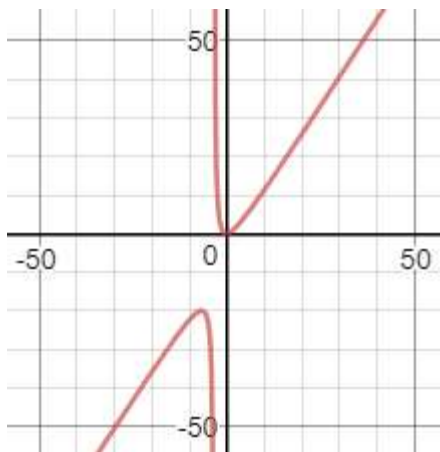
5. Find the value of $\lim_{x \rightarrow \infty} \frac{2x+x}{3x^2+5x}$ using a graph.

Notes

Guided Example

Find the value of $\lim_{x \rightarrow \infty} \frac{3x^2 + x}{2x + 7}$ using a graph.

Solution Start by making a graph of the expression. Make sure the window is large enough to see how the graph behaves on the right.



Following the graph farther and farther to the right, it appears that the graph is growing. This means as x gets more and more positive, the outputs are getting larger and larger. Since the values do not approach a number, we say that the limit does not exist. Since it does this by getting positive and big, we write

$$\lim_{x \rightarrow \infty} \frac{3x^2 + x}{2x + 7} \approx 0$$

Practice

6. Find the value of $\lim_{x \rightarrow \infty} \frac{x^4 + 1}{3x + 6}$ using a graph.

Notes

Question 2 – How do you evaluate limits at infinity using algebra?

Key Terms

Limit at infinity

Summary

If the limit you are examining is a rational polynomial (a fraction of polynomials), you can compute the value of the limit exactly. This is done by taking advantage of the fact that

$$\lim_{x \rightarrow \infty} \frac{1}{x^n} = 0 \quad \text{or} \quad \lim_{x \rightarrow -\infty} \frac{1}{x^n} = 0$$

for any positive real number n . To do this, we need to transform the rational expression by locating the highest power in the denominator on a variable. Then divide each term in the rational expression by x to that power and simplify. Once simplified, we can utilize the limits above to determine which terms are small as x grow more and more positive or more and more negative.

Notes

Guided ExamplePractice

Find the value of $\lim_{x \rightarrow \infty} \frac{6x^2 + 7}{3x^2 + x}$ algebraically.

Solution The highest power on a variable in the denominator is 2. Divide each term in the expression by x to that power or x^2 :

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{6x^2 + 7}{3x^2 + x} &= \lim_{x \rightarrow \infty} \frac{\frac{6x^2}{x^2} + \frac{7}{x^2}}{\frac{3x^2}{x^2} + \frac{x}{x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{6 + \frac{7}{x^2}}{3 + \frac{1}{x}} \end{aligned}$$

Reduce each fraction

The terms $\frac{7}{x^2}$ and $\frac{1}{x}$ become small as x gets large. This means the numerator will get closer and closer to 6 while the denominator gets closer and closer to 3. Putting these two together gives

$$\lim_{x \rightarrow \infty} \frac{6x^2 + 7}{3x^2 + x} = \frac{6}{3} = 2$$

1. Find the value of $\lim_{x \rightarrow \infty} \frac{5x^2 + x}{3x^2 + 2x}$

algebraically

Guided ExamplePractice

Find the value of $\lim_{x \rightarrow \infty} \frac{2x + 7}{3x^2 + x}$ algebraically.

Solution The highest power on a variable in the denominator is 2. Divide each term in the expression by x to that power or x^2 :

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{2x + 7}{3x^2 + x} &= \lim_{x \rightarrow \infty} \frac{\frac{2x}{x^2} + \frac{7}{x^2}}{\frac{3x^2}{x^2} + \frac{x}{x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{2}{x} + \frac{7}{x^2}}{3 + \frac{1}{x}} \end{aligned}$$

Reduce each fraction

2. Find the value of $\lim_{x \rightarrow \infty} \frac{2x + x}{3x^2 + 5x}$

algebraically.

<p>The terms $\frac{2}{x}$, $\frac{7}{x^2}$ and $\frac{1}{x}$ become small as x gets negative and large. This means the numerator will get closer and closer to 0 while the denominator gets closer and closer to 3. Putting these two together gives</p> $\lim_{x \rightarrow -\infty} \frac{2x+7}{3x^2+x} = \frac{0}{3} = 0$	
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Guided Example

Practice

<p>Find the value of $\lim_{x \rightarrow \infty} \frac{3x^2+x}{2x+7}$ algebraically.</p> <p>Solution The highest power on a variable in the denominator is 1. Divide each term in the expression by x to that power or x:</p> $\begin{aligned} \lim_{x \rightarrow \infty} \frac{3x^2+x}{2x+7} &= \lim_{x \rightarrow \infty} \frac{\frac{3x^2}{x} + \frac{x}{x}}{\frac{2x}{x} + \frac{7}{x}} \\ &= \lim_{x \rightarrow \infty} \frac{3x + \frac{1}{x}}{2 + \frac{7}{x}} \end{aligned}$ <p>The terms $\frac{7}{x}$ and $\frac{1}{x}$ become small as x gets large. However, the term $3x$ will get large as x grows. This means the numerator will grow while the denominator gets closer and closer to 2. When the numerator grows like this, the entire fraction will grow so the limit does not exist. Since the fraction grows more and more positive, we write</p> $\lim_{x \rightarrow \infty} \frac{3x^2+x}{2x+7} = +\infty$	<p>3. Find the value of $\lim_{x \rightarrow \infty} \frac{x^4+1}{3x+6}$ algebraically.</p>
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Section 10.5 Continuous Functions

Question 1 – What does a continuous function look like?

Question 2 – How is a limit related to a continuous function?

Question 1 – What does a continuous function look like?

Key Terms

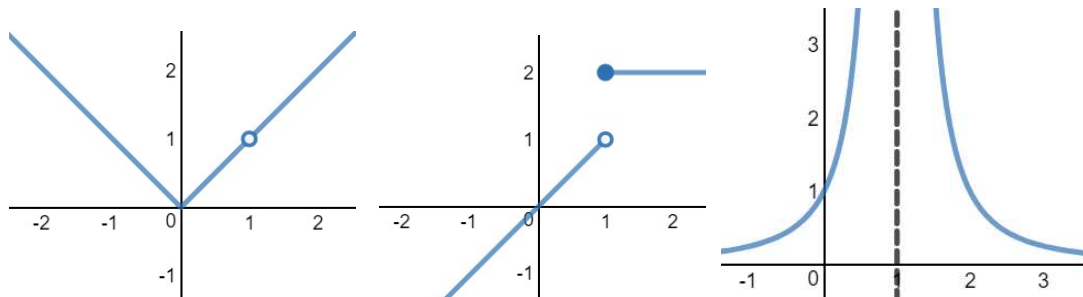
Continuous at a point

Discontinuity

Continuous function

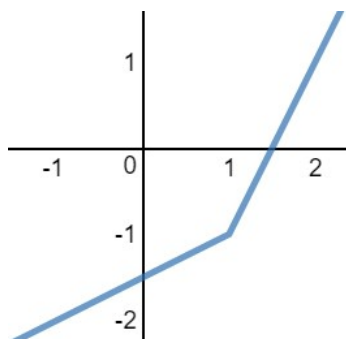
Summary

In a graphical sense, a continuous function is a function that can be drawn without lifting your writing utensil off the paper. Discontinuities are where you must lift your writing utensil up. Polynomials, such as lines and parabolas, are continuous everywhere. Piecewise functions may or may not be continuous even if the pieces themselves are polynomials.



Each of the functions above have a discontinuity at $x = 1$. The graph on the left has a hole because it is not defined there. The graph in the middle has a gap at $x = 1$. The graph on the right has a vertical asymptote at $x = 1$ that means it is not defined at $x = 1$.

Kinks in graphs do not cause them to be discontinuous by themselves.



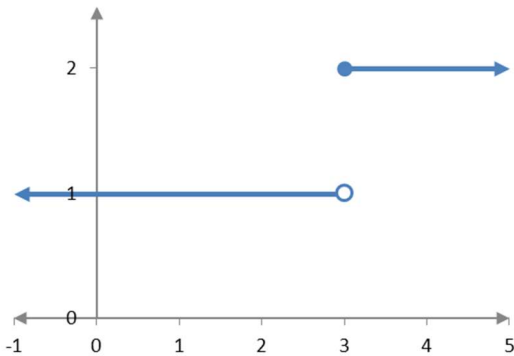
The graph above is continuous everywhere despite the kink in its appearance.

Notes

Guided Example

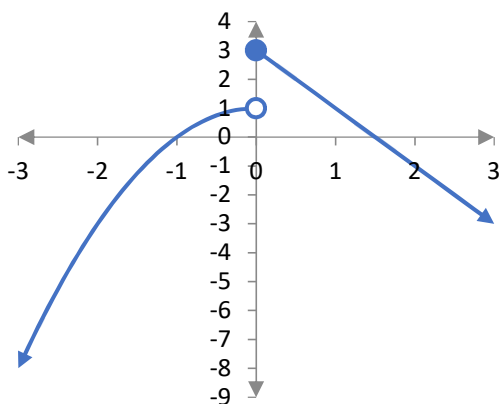
Where is the function not continuous?

a.



Solution To be continuous, you should be able to draw the function without lifting your drawing utensil. In the case of this function, you would need to lift your utensil at $x = 3$ to jump to the other piece. Because of this, the function is discontinuous at $x = 3$.

b.

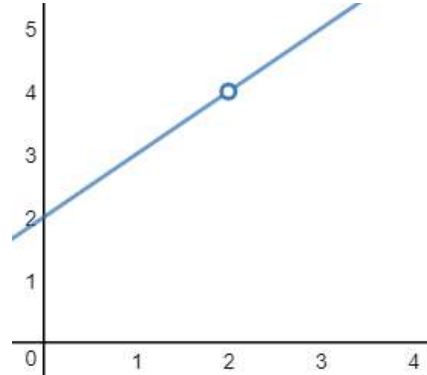


Solution To draw this function, you would need to lift your drawing utensil at $x = 0$ to jump to the other piece of the function. Therefore, the function is discontinuous at $x = 0$.

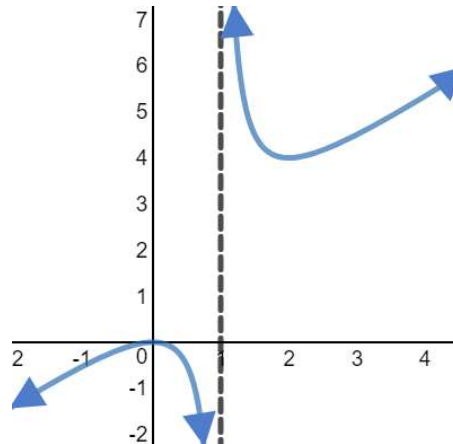
Practice

1. Where is the function not continuous?

a.



b.



Question 2 – How is a limit related to a continuous function?

Key Terms

Continuous at a point Discontinuity

Continuous function

Summary

The definition of a continuous function is related to limits. In order for a function $f(x)$ to be continuous at some point $x = a$, three conditions must be met.

1. $f(a)$ is defined.
2. $\lim_{x \rightarrow a} f(x)$ exists.
3. $\lim_{x \rightarrow a} f(x) = f(a)$

To utilize this definition, you must compute the limits from the left and right (to make sure the limit exists) and compare those values to $f(a)$. If the left-hand limit, the right-hand limit, and $f(a)$ are all the same, then the function is continuous at $x = a$.

Notes

Guided ExamplePractice

Compute the limits to determine whether the function

$$f(x) = \begin{cases} -x + 4 & \text{if } x < 3 \\ x - 2 & \text{if } x > 3 \\ 0 & \text{if } x = 3 \end{cases}$$

to determine where $f(x)$ is continuous at $x = 3$.

a. $\lim_{x \rightarrow 3^-} f(x)$

Solution To do the left-hand limit, we use the portion of the graph that is valid for $x < 3$ and put that in place of $f(x)$. Then we can evaluate the limit by setting $x = 3$ in that formula.

$$\begin{aligned} \lim_{x \rightarrow 3^-} f(x) &= \lim_{x \rightarrow 3^-} -x + 4 \\ &= 1 \end{aligned}$$

b. $\lim_{x \rightarrow 3^+} f(x)$

Solution For the right-hand limit, use the portion of the formula valid for $x > 3$ and put in the point:

$$\begin{aligned} \lim_{x \rightarrow 3^+} f(x) &= \lim_{x \rightarrow 3^+} x - 2 \\ &= 1 \end{aligned}$$

c. Is the function continuous at $x = 3$?

Solution For the function to be continuous at $x = 3$, the left-hand limit, the right-hand limit, and $f(3)$ must be equal. Since they are not, the function is discontinuous at $x = 3$.

1. Compute the limits to determine whether the function

$$f(x) = \begin{cases} -x^2 + 1 & \text{if } x < 0 \\ -2x + 3 & \text{if } x \geq 0 \end{cases}$$

to determine where $f(x)$ is continuous at $x = 0$.

a. $\lim_{x \rightarrow 0^-} f(x)$

b. $\lim_{x \rightarrow 0^+} f(x)$

c. Is the function continuous at $x = 0$?

Guided Example

Practice

A state in the Midwest has a graduated state income tax given by

$$T(x) = \begin{cases} 0.01x & \text{if } 0 \leq x \leq 10,000 \\ 0.015x + b & \text{if } x > 10,000 \end{cases}$$

where x is the taxable income of an individual taxpayer and b is a constant.

a. $\lim_{x \rightarrow 10,000^-} T(x)$

Solution To evaluate the left-hand limit, use the top portion of the piecewise function that is valid for $0 \leq x \leq 10,000$:

$$\lim_{x \rightarrow 10,000^-} T(x) = \lim_{x \rightarrow 10,000^-} 0.01x = 0.01(10,000) = 100$$

b. $\lim_{x \rightarrow 10,000^+} T(x)$

Solution For the right-hand limit, use the portion of the piecewise function that is valid for $x > 10,000$:

$$\begin{aligned} \lim_{x \rightarrow 10,000^+} T(x) &= \lim_{x \rightarrow 10,000^+} 0.015x + b \\ &= 0.015(10,000) + b \\ &= 150 + b \end{aligned}$$

c. What value of b would make this function continuous?

Solution For the function to be continuous, the left and right-hand limits must be equal. Set the limits equal and solve for b :

$$\begin{aligned} 100 &= 150 + b \\ -50 &= b \end{aligned}$$

Also note that $f(10,000) = 0.01(10,000) = 100$.

2. A state in the East has a graduated state income tax given by

$$T(x) = \begin{cases} 0.03x & \text{if } 0 \leq x \leq 15,000 \\ 0.045x + b & \text{if } x > 15,000 \end{cases}$$

where x is the taxable income of an individual taxpayer and b is a constant.

a. $\lim_{x \rightarrow 15,000^-} T(x)$

b. $\lim_{x \rightarrow 15,000^+} T(x)$

c. What value of b would make this function continuous?

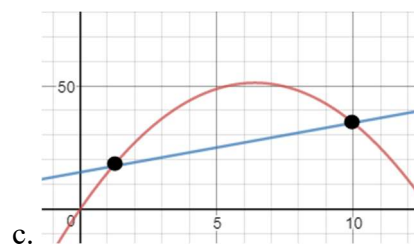
Chapter 10 Answers

Section 10.1

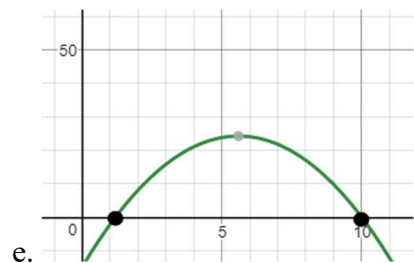
Question 1 1) Yes, 2) No, 3) a. Yes, b. Yes

Question 2 4) a. 9.75, 3.5, b. 3.75, 7.5, c. 4.8, d. 8

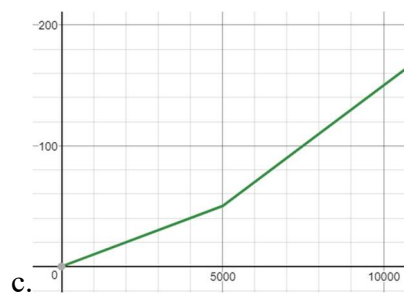
5) a. $16Q - 1.25Q^2$, b. 10, 1.2,



d. $-1.25Q^2 + 14Q - 15$



Question 3 6) a. 40, b. 90



$$7) \text{ a. } 8, \text{ b. } 70, \text{ c. } 260, T(x) = \begin{cases} .01x & \text{if } 0 \leq x \leq 1000 \\ .03x - 20 & \text{if } 1000 < x \leq 6000 \\ .05x - 160 & \text{if } x > 6000 \end{cases}$$

Section 10.2

Question 1 1) $\frac{1}{4}$

2) Does not exist since the left-hand limit is $+\infty$ and the right-hand limit is $-\infty$.

3) a. 50, b. 50, c. 50

Question 2 1) a. 3, b. 3, c. 3

2) a. 1, b. 3, c. does not exist

Section 10.3

Question 1 1) 6, 2) $\frac{1}{16}$, 3) $\frac{1}{4}$

Question 2 1) 8, 2) 4, 3) $2x + 2$, 4) $\frac{-1}{2x^2}$

Section 10.4

Question 1 1) 1.666, 2) 0, 3) ∞ , 4) 1.666, 5) 0, 6) $+\infty$

Question 2 1) $5/3$, 2) 0, 3) $+\infty$

Section 10.5

Question 1 1) a. $x = 2$, b. $x = 1$

Question 2 1) a. 1, b. 3, c. not continuous, 2) a. 450, b. $675 + b$, c. -225