Section 11.1 Average Rate of Change

Question 1 – How do you calculate the average rate of change from a table?

Question 2 – How do you calculate the average rate of change from a function?

Question 1 – How do you calculate the average rate of change from a table?

Key Terms

Average rate of change

Delta notation

Summary

The average rate of change of a function f with respect to x from $x = a$ to $x = b$ is defined as

$$\begin{matrix}\text{Average rate of change of } f \\ \text{with respect to } x \text{ over } [a, b]\end{matrix} = \frac{\text{Change in } f}{\text{Change in } x}$$

The symbol Δ is often used to represent change in. In this case, the can shorten the definition to

$$\begin{matrix}\text{Average rate of change of } f \\ \text{with respect to } x \text{ over } [a, b]\end{matrix} = \frac{\Delta f}{\Delta x}$$

This ratio compares how one quantity is changing when another quantity is changing. When you see the words "with respect to", this indicates that the denominator of the average rate has to do with a change in that quantity.

The units on the average rate of change are determined by the units on the quantities. In general, the units are the units on the numerator divided by the units on the denominator. For instance, If we wanted to calculate the average rate of change of cost (in dollars) with respect to a change in quantity (toasters), the units on the rate would be

$$\frac{\text{dollars}}{\text{toaster}} \text{ or dollars per toaster}$$

Notes

Guided Example

Over the last few years, the amount of money spent on online advertising has increased. The table shows the amount spent on online advertising, in millions of dollars, over the period 2006 to 2011.

Year	Online Ad Spending (millions $)
2006	259
2007	370
2008	508
2009	650
2010	760
2011	857

a. Find the average rate of change of spending with respect to time from 2006 to 2008.

Solution To find the average rate of change of spending with respect to time over 2006 to 2008, we need to calculate the change in online spending and the change in time over this time period:

$$\frac{\Delta \text{Ad Spending}}{\Delta \text{time}} = \frac{508 - 259}{2008 - 2006} \frac{\text{million dollars}}{\text{years}}$$

$$= \frac{249}{2}$$

$$= 124.5 \text{ million dollars per year}$$

Practice

1. Capital expenditures are payments that are used to purchase or upgrade physical assets. The table gives the amount of capital expenditures at Verizon from 2006 to 2011.

Year	Capital Expenditures (billions $)
2006	17.1
2007	17.5
2008	17.2
2009	16.9
2010	16.5
2011	16.2

a. Find the average rate of change of capital expenditures with respect to time from 2006 to 2008.

b. Find the average rate of change of spending with respect to time from 2008 to 2011. **Solution** We calculate the average rate the same way, but from 2008 to 2011. $$\frac{\Delta\text{Ad Spending}}{\Delta\text{time}} = \frac{857-508}{2011-2008}\frac{\text{million dollars}}{\text{years}}$$ $$= \frac{349}{3}$$ $$\approx 116.3 \text{ million dollars per year}$$	b. Find the average rate of change of capital expenditures with respect to time from 2008 to 2011.
c. What do the numbers in parts a and b tell you about online ad spending? **Solution** Online add spending increased, but the rate at which spending is changing is getting smaller over time.	c. What do the rates in parts a and b tell you about capital expenditures at Verizon from 2006 to 2011?

Guided Example

The table gives the annual revenue from Verizon’s wireless business and the number of wireless connections that yields this revenue.

Year	Wireless Revenue (billions $)	Wireless Connections (millions)
2006	38.0	59.1
2007	43.9	65.7
2008	49.3	72.1
2009	60.3	96.5
2010	63.4	102.2
2011	70.2	107.8

a. Find the average rate of change of wireless revenue with respect to time from 2006 to 2008.

Solution Use the table to find the data for wireless revenue in 2006 and 2008. The average rate is

$$\frac{\Delta\text{revenue}}{\Delta\text{time}} = \frac{49.3-38.0}{2008-2006}\,\frac{\text{billion dollars}}{\text{years}}$$

$$= \frac{11.3}{2}$$

$$= 5.65 \text{ billion dollars per year}$$

Practice

2. The table gives the annual revenue from Verizon’s wireless business and the number of wireless connections that yields this revenue.

Year	Wireless Revenue (billions $)	Wireless Connections (millions)
2006	38.0	59.1
2007	43.9	65.7
2008	49.3	72.1
2009	60.3	96.5
2010	63.4	102.2
2011	70.2	107.8

a. Find the average rate of change of wireless revenue with respect to time from 2008 to 2011.

b. Find the average rate of change of wireless connections with respect to time from 2006 to 2008.

Solution Use the table to find the data for wireless connections in 2006 and 2008. The average rate is

$$\frac{\Delta connections}{\Delta \text{time}} = \frac{72.1 - 59.1}{2008 - 2006} \frac{\text{million connections}}{\text{years}}$$

$$= \frac{13.0}{2}$$

$$= 6.5 \text{ million connections per year}$$

c. Find the average rate of change of wireless revenue with respect to wireless connections from 2006 to 2008.

Solution Now we need to locate the wireless revenue and connections in 2006 and 2008.

$$\frac{\Delta \text{revenue}}{\Delta \text{connections}} = \frac{49.3 - 38.0}{72.1 - 59.1} \frac{\text{billion dollars}}{\text{million connections}}$$

$$= \frac{11.3}{13}$$

$$= 0.869$$

The units on this number are obtained by dividing billions by millions to give thousands. So the average rate is 0.869 thousand dollars per connection. This could also be written as 869 dollars per connection. This means that each additional connection from 2006 to 2008 increased annual revenue by $869 (or about $72 per month).

b. Find the average rate of change of wireless connections with respect to time from 2008 to 2011.

c. Find the average rate of change of wireless revenue with respect to wireless connections from 2008 to 2011.

Question 2 – How do you calculate the average rate of change from a function?

Key Terms

Average rate of change

Summary

When the average rate of change is calculated from a function's formula, we calculate the change in the numerator using the functions formula. The average rate of change of f with respect to x from $x = a$ to $x = b$ is

$$\frac{\Delta f}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

For example, if we want to calculate the average rate of change of $f(x) = x^2$ from $x = 2$ to $x = 4$, we would first calculate

$$f(2) = 2^2 = 4$$
$$f(4) = 4^2 = 16$$

And then calculate the rate,

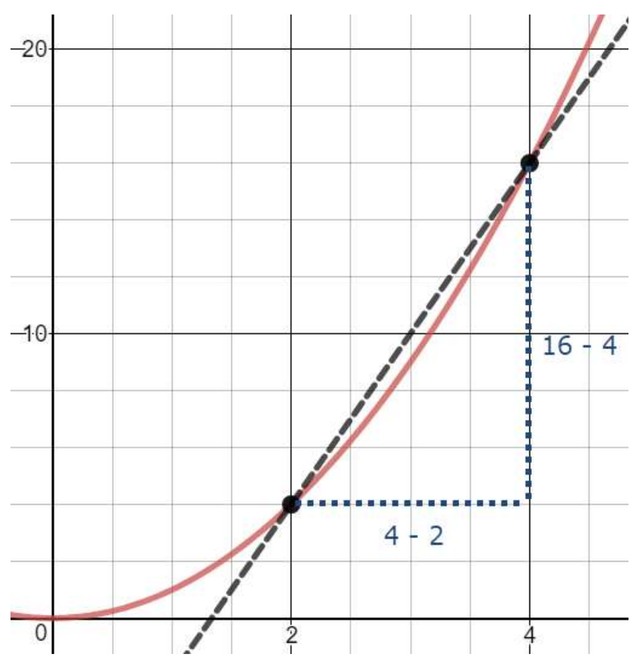
$$\frac{\Delta f}{\Delta x} = \frac{16 - 4}{4 - 2} = 6$$

This can be interpreted graphically by drawing a line through the points on the function.

The black dashed line through the points (2, 4) and (4, 16) is called a secant line through the points. If we calculate the slope of the secant line, we get

$$m_{\text{secant}} = \frac{16 - 4}{4 - 2} = 6$$

This is the exact same ratio that we calculated for the average rate of change between $x = 2$ and $x = 4$. This leads us to describe the average rate of change of f with respect to x from $x = a$ to $x = b$ as the slope of a secant line through $x = a$ and $x = b$.



Notes

Guided Example

Based on data from 2009 through 2011, the amount of money spent on research at Johnson and Johnson is

$$R(t) = 4729.3\left(1.042^{t}\right) \text{ million dollars}$$

where t is the number of years since 2000.

Find the average rate of change of research money with respect to time from 2000 to 2009.

Solution To compute the average rate of change of research money (represented by R(t)) with respect to time from 2000 to 2009, we need to find

$$\frac{\Delta R}{\Delta t} = \frac{R(9) - R(0)}{9 - 0}$$

Where 0 and 9 are used since t represents years after 2000. Start by finding the function values,

$$R(0) = 4729.3\left(1.042^{t0}\right) = 4729.3 \text{ million dollars}$$

$$R(9) = 4729.3\left(1.042^{9}\right) \approx 6848.7 \text{ million dollars}$$

This gives us the average rate of change

$$\begin{aligned}\frac{\Delta R}{\Delta t} &= \frac{6848.7 - 4729.3}{9 - 0} \frac{\text{million dollars}}{\text{year}} \\ &= \frac{2119.4}{9} \\ &\approx 235.5 \text{ million dollars per year}\end{aligned}$$

This means that from 2000 to 2009, research spending grew by an average of 235.5 each year.

Practice

Based on data from 2009 through 2011, the amount of money spent on research at Johnson and Johnson is

$$R(t) = 4729.3\left(1.042^{t}\right) \text{ million dollars}$$

where t is the number of years since 2000.

Find the average rate of change of research money with respect to time from 2011 to 2015.

Section 11.2 Instantaneous Rate of Change

Question 1 – How do you estimate the instantaneous rate of change?

Question 2 – How do you compute the instantaneous rate of change using a limit?

Question 1 – How do you estimate the instantaneous rate of change?

<u>Key Terms</u>

Estimate

Instantaneous rate of change

<u>Summary</u>

The instantaneous rate of change is almost identical to the average rate of change. In both cases, you are comparing a change in the dependent variable with respect to a change in the independent variable. The main difference is the size of the change in the independent variable (the denominator in the rate). For an average rate of change, the change in the independent variable can be any size. However, for an instantaneous rate of change we want the change in the independent variable to be as small as possible.

If a function is given by a table, we may be limited by the degree by which the change in the independent variable may be made. In this case, we typically utilize the data point at which we want the rate (or the closest data available) and the next closest data value to estimate the instantaneous rate of change.

<u>Notes</u>

Guided Example

The table on the right records the level of the Dow Jones Industrial Average (DJIA) in points minutes after 1PM on May 6, 2010. Use the table to find the rates below.

Minutes after 1PM	DJIA (points)
1.7	10800
65.0	10700
90.0	10600
98.3	10500
103.3	10400
105.0	10300

a. Find the average rate of change of the Dow Jones Industrial Average with respect to time from 1.7 minutes after 1PM to 105.0 minutes after 1PM.

Solution Use the function name DJIA for the outputs in the second column of the table. To find the average rate of change over [1.7, 105.0], calculate

$$\frac{DJIA(105.0)-DJIA(1.7)}{105.0-1.7}=\frac{10300-10800}{105.0-1.7}$$

$$\approx -4.84$$

The DJIA is decreasing by an average of 4.84 points per minute.

b. Estimate the instantaneous rate of change of the Dow Jones Industrial Average at 105 minutes after 1PM.

Solution To find the instantaneous rate, we need to calculate the average rate over a very small interval of time close to 105.0 minutes. The best estimate is obtained over the interval [103.3, 105,0]. The average rate of change of DJIA over [103.3, 105.0] is

$$\frac{DJIA(105.0)-DJIA(103.3)}{105.0-103.3}=\frac{10300-10400}{105.0-103.3}$$

$$\approx -58.82$$

At 105.0 minutes, DJIA is decreasing by approximately -58.82 points per minute.

Practice

1. The table to the right describes the number of barrels of beer produced annually (in millions) by the Boston Beer Company at several different staffing levels.

a. Find the average rate of change of the amount of beer with respect to the number of employees from 200 employees to 365 employees.

Number of Employees	Amount of Beer (millions of barrels)
200	0.498
250	0.709
300	0.939
360	1.228
363	1.243
365	1.253

b. Estimate the instantaneous rate of change if the amount of beer when staffing is 365 employees.

Question 2 – How do you compute the instantaneous rate of change using a limit?

<u>Key Terms</u>

Instantaneous rate of change

<u>Summary</u>

When a function is given by a formula, we can do more than estimate the instantaneous rate of change from two adjacent data values. We can pick any two values separated by an arbitrary amount h. If one of the points is $(a, f(a))$, then another point h units away from a would be $(a + h, f(a + h))$. The average rate of change would be

$$\frac{f(a+h)-f(a)}{a+h-a}=\frac{f(a+h)-f(a)}{h}$$

where h is the change in the value of the independent variable.

To make this into an instantaneous rate of change, we need to make this value as small as possible. This is accomplished by using a limit and letting h approach 0:

$$\begin{array}{c}\text{Instantaneous Rate of Change}\\ \text{of } f(x) \text{ with respect to x at } x=a\end{array}=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

By evaluating this limit, we obtain the exact value of the instantaneous rate of change.

<u>Notes</u>

Guided Example

The demand for a particular product is given by

$$D(p) = -2p^2 - 2p + 400 \quad \text{items}$$

where p is the unit price in dollars. Find the instantaneous rate of change of demand at a price of \$5.

Solution To calculate the instantaneous rate of change at $p = 5$, we need to evaluate the limit

$$\lim_{h \to 0} \frac{D(5+h) - D(5)}{h}$$

Start by finding the two function values:

$$D(5) = -2(5)^2 - 2(5) + 400 = 340$$

and

$$\begin{aligned} D(5+h) &= -2\overbrace{(5+h)^2}^{25+10h+h^2} - 2(5+h) + 400 \\ &= -50 - 20h - 2h^2 - 10 - 2h + 400 \\ &= -2h^2 - 22h + 340 \end{aligned}$$

Now combine these values in the limit:

$$\begin{aligned} \lim_{h \to 0} \frac{-2h^2 - 22h + \cancel{340} - \cancel{340}}{h} &= \lim_{h \to 0} \frac{\cancel{h}(-2h - 22)}{\cancel{h}} \\ &= \lim_{h \to 0} \; -2h - 22 \\ &= -22 \end{aligned}$$

At a price of \$5, the demand is decreasing by 22 items per dollar.

Practice

1. The demand for a particular product is given by

$$D(p) = -4p^2 - 4p + 700 \quad \text{items}$$

where p is the unit price in dollars. Find the instantaneous rate of change of demand at a price of \$6.

Section 11.3 The Derivative at a Point

Question 1 – What is a derivative?

Question 2 – How do you compute the derivative at a point using a limit?

Question 3 - How can you use a tangent line to forecast function values?

Question 4 - What does the derivative at a point tell you about a function?

Question 1 – What is a derivative?

Key Terms

Derivative

Tangent line

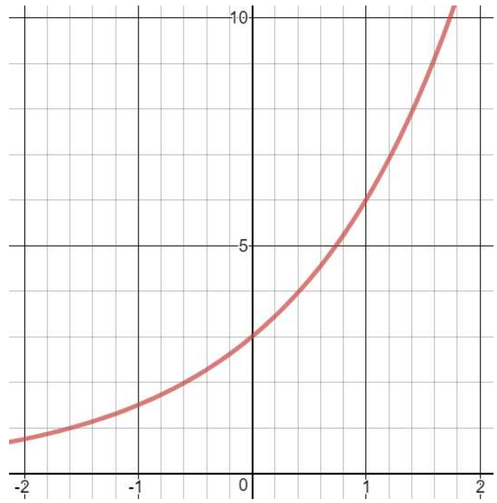
Summary

The derivative of a function $f(x)$ at a particular point $x = a$ is the slope of the tangent line to the function at $x = a$. To estimate the value of the derivative of f at a (denoted by $f'(a)$, draw the tangent line on a graph and then estimate its slope using any grids or scales available on the graph.

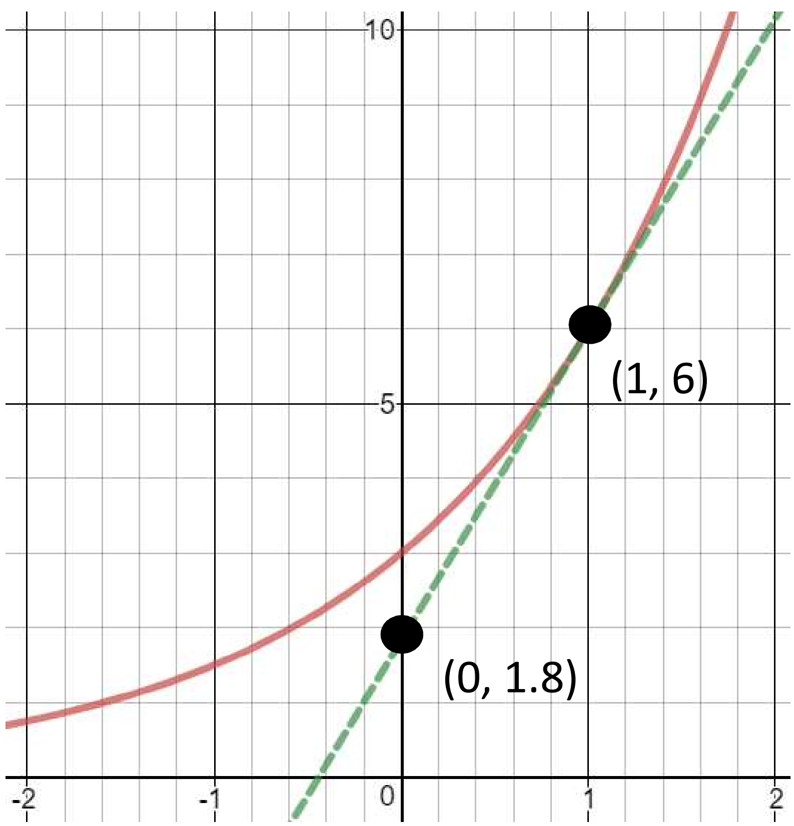
Notes

Guided Example

For the function $f(x)$ graphed below, estimate $f'(1)$



Solution Draw the tangent line on the graph and locate two points on the tangent line.

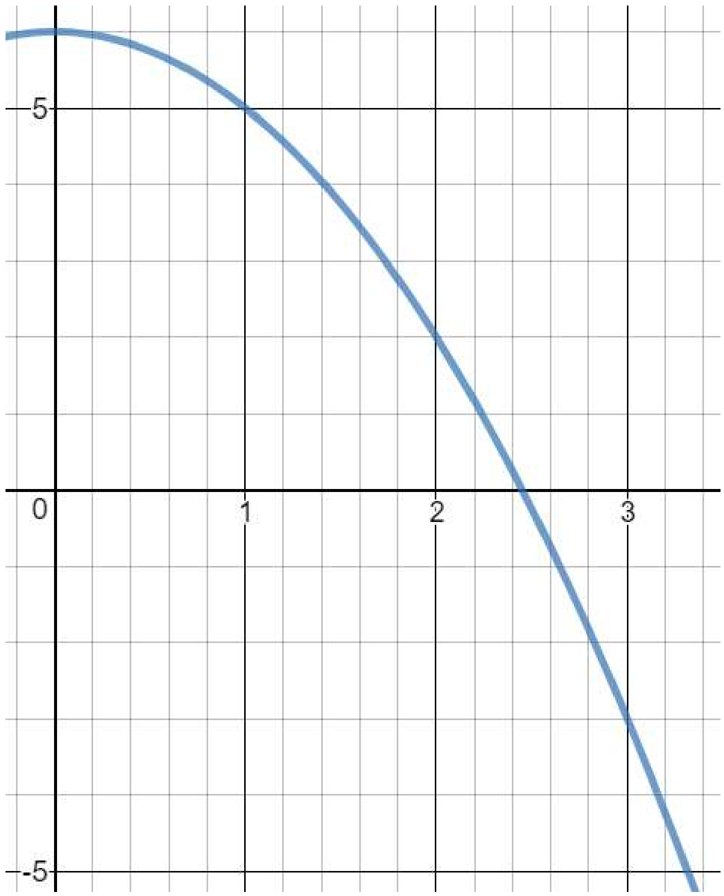


Calculate the slope from these two points to give an estimate of the value of the derivative,

$$f'(1) \approx \frac{6-1.8}{1.0} \approx 4.2$$

Practice

1. For the function $g(x)$ graphed below, estimate $g'(2)$



Question 2 – How do you compute the derivative at a point using a limit?

Key Terms

Limit definition of derivative at a point

Summary

The slope of the tangent line at a point may be calculated using the instantaneous rate of change at the point. For a function $f(x)$ at a point $x = a$, the derivative is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists. Evaluating this limit may require some of the tools from Section 10.3.

Notes

<u>Guided Example</u>

Suppose $f(x) = -2x - 3$. Use the definition of the derivative to compute $f'(8)$.

Solution The definition of the derivative at a point $a = 8$ is,

$$f'(8) = \lim_{h \to 0} \frac{f(8+h) - f(8)}{h} \ .$$

To apply this definition, we need to calculate the two function values in the numerator of the difference quotient.

$$f(8) = -2(8) - 3 = -19$$

$$f(8+h) = -2(8+h) - 3 = -19 - 2h$$

This is put into the definition to give

$$\begin{aligned} f'(8) &= \lim_{h \to 0} \frac{\cancel{-19} - 2h - (\cancel{-19})}{h} \\ &= \lim_{h \to 0} \frac{-2h}{h} \\ &= \lim_{h \to 0} -2 \\ &= -2 \end{aligned}$$

Since there is no h in the final limit, h approaching zero does nothing.

<u>Practice</u>

1. Suppose $f(x) = 5x - 3$. Use the definition of the derivative to compute $f'(-5)$.

Guided Example	Practice
Suppose $f(x) = 3x^2 - 4x + 1$. Use the definition of the derivative to compute $f'(2)$. **Solution** The definition of the derivative at a point *a* = 2 is, $$f'(2) = \lim_{h\to 0}\frac{f(2+h) - f(2)}{h}\quad .$$ To apply this definition, we need to calculate the two function values in the numerator of the difference quotient. $$f(2) = 3(2)^2 - 4(2) + 1 = 5$$ $$\begin{aligned} f(2+h) &= 3(2+h)^2 - 4(2+h) + 1 \\ &= 12 + 12h + 3h^2 - 8 - 4h + 1 \\ &= 3h^2 + 8h + 5 \end{aligned}$$ Now put the function values into the difference quotient: $$\begin{aligned} f'(2) &= \lim_{h\to 0}\frac{3h^2 + 8h \cancel{+5} \cancel{-5}}{h} \\ &= \lim_{h\to 0}\frac{\cancel{h}(3h+8)}{\cancel{h}} \\ &= \lim_{h\to 0} 3h + 8 \\ &= 8 \end{aligned}$$	Suppose $f(x) = 2x^2 + x - 2$. Use the definition of the derivative to compute $f'(-1)$.

Question 3 – How can you use a tangent line to forecast function values?

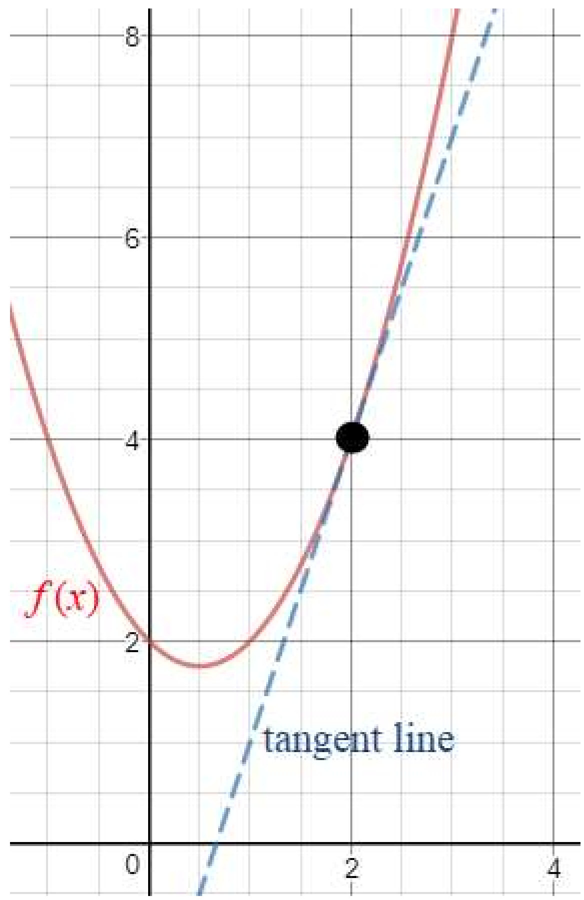
Key Terms

Tangent line

Forecast

Summary

The tangent line to a function can be used as a quick way to forecast the value of the function at some point. To find the equation of the tangent line, we start from $y = mx + b$. The slope of the tangent line, m, is found by evaluating the derivative at a given point. The vertical intercept, b, is found by substituting the point at which the line is tangent into the original function.



Notes

Guided Example | Practice

Guided Example	Practice

For the function $f(x) = x^2 - x + 2$, answer each of the questions below.

a. Find the equation of the tangent line at $x = 2$.

Solution To find the equation of the tangent line, we need to find the point on the function it will pass through and the slope of the tangent line at that point.

To find the point on the function, simply substitute $x = 2$ into the function.

$$f(2) = 2^2 - 2 + 2 = 4$$

The tangent line will have to pass through the point (2, 4).

To find the slope of the tangent line, we need to find the value of the derivative at $x = 2$:

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

where

$$f(2) = (2)^2 - (2) + 2 = 4$$

$$\begin{aligned} f(2+h) &= (2+h)^2 - (2+h) + 2 \\ &= 4 + 4h + h^2 - 2 - h + 2 \\ &= h^2 + 3h + 4 \end{aligned}$$

Using these function values in the definition of the derivative gives

$$\begin{aligned} f'(2) &= \lim_{h \to 0} \frac{h^2 + 3h \cancel{+4} \cancel{-4}}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(h+3)}{\cancel{h}} \\ &= 3 \end{aligned}$$

1. For the function $f(x) = x^2 - 2x + 3$, answer each of the questions below.

a. Find the equation of the tangent line at $x = 1$.

Based on this information, the tangent line has slope 3 and passes through (2, 4). The equation is found by starting with the equation of a line, $y = mx + b$. We know the slope is 3 so we can let $m = 3$ to yield $$y = 3x + b$$ You can find the value of b by substituting (2, 4) into this equation. $$4 = 3(2) + b$$ $$-2 = b$$ Using this value, we find the equation of the tangent line, $$y = 3x - 2$$ b. Use the tangent line to forecast the value of the function at $x = 2.5$. **Solution** We can forecast values on the function using the tangent line. To do this, substitute x = 4 into the tangent line: $$y = 3(2.5) - 2 = 5.5$$	b. Use the tangent line to forecast the value of the function at $x = 1.75$.

Question 4 – What does the derivative at a point tell you about a function?

Key Terms

Units

Dependent variable

Independent variable

Summary

When you evaluate a derivative in the context of an application, you get a number that indicates how fast two quantities are changing with respect to each other. The units on the independent and dependent variable determine the units on the value of the derivative.

$$\text{units on the derivative} = \frac{\text{units of the dependent variable}}{\text{units on the independent variable}}$$

For instance, suppose we have a demand function $D(p)$. Since this is a demand function, it relates the unit price (in this case in dollars) to the number of units demanded at this price level. The derivative of this demand function would have units

$$\frac{\text{units}}{\text{dollars}}$$

Using this number, we can examine how an increase in price of 1 dollar will impact consumer demand.

Notes

Guided Example | Practice

Guided Example	Practice
The profit (in thousands of dollars) for selling x hundred units of compressors is $$P(x) = -4x^2 + 160x - 1000$$ Find and interpret $P'(10)$. **Solution** Start by writing out the definition for this derivative: $$P'(10) = \lim_{h \to 0} \frac{P(10+h) - P(10)}{h}$$ where $$P(10) = -4(10)^2 + 160(10) - 1000 = 200$$ $$\begin{aligned} P(10+h) &= -4(10+h)^2 + 160(10+h) - 1000 \\ &= -4(100 + 20h + h^2) + 160(10+h) - 1000 \\ &= -400 - 80h - 4h^2 + 1600 + 160h - 1000 \\ &= -4h^2 + 80h + 200 \end{aligned}$$ Using these function values in the definition of the derivative gives $$\begin{aligned} P'(10) &= \lim_{h \to 0} \frac{-4h^2 + 80h \cancel{+200} \cancel{-200}}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(-4h + 80)}{\cancel{h}} \\ &= 80 \end{aligned}$$ This number does not have much meaning without units attached to it. Since it represents a slope on the graph of $P(x)$, the corresponding units are the units of the dependent variable divided by the units of the independent variable. $$\frac{\text{thousands of dollars}}{\text{hundreds of compressors}}$$	1. The profit (in thousands of dollars) for selling x hundred units of calculators is $$P(x) = -5x^2 + 80x - 100$$ Find and interpret $P'(7)$.

Simplifying thousands divided by hundreds to give hundreds, these units become $$\frac{\text{tens of dollars}}{\text{compressors}}$$ Including this in our derivative gives $$P'(10) = 80\,\frac{\text{tens of dollars}}{\text{compressors}}$$ This means that increasing the number of compressors by 1 unit will increase the profit by 800 dollars. Another way to say this is to indicate that producing the 1001st compressor will increase profit by \$800.	

Section 11.4 The Derivative Function

Question 1 – What is a derivative function?

Question 2 – How do you calculate the derivative of a function from the definition?

Question 3 - What are the derivatives of some basic functions (linear, polynomial, power, exponential, and logarithmic)?

Question 1 – What is a derivative function?

<u>Key Terms</u>

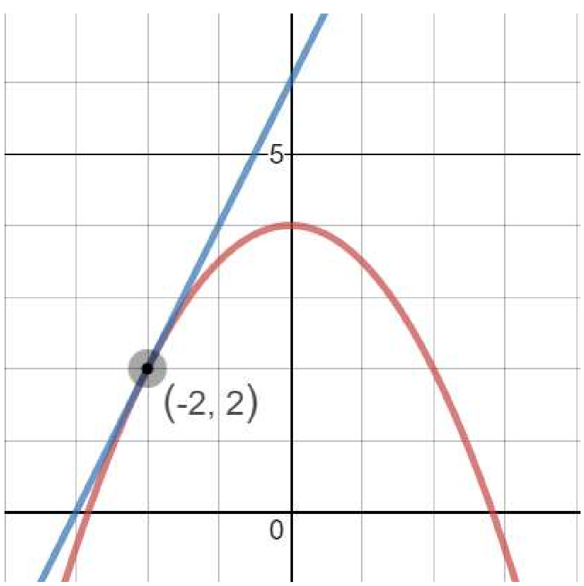
Derivative

Function

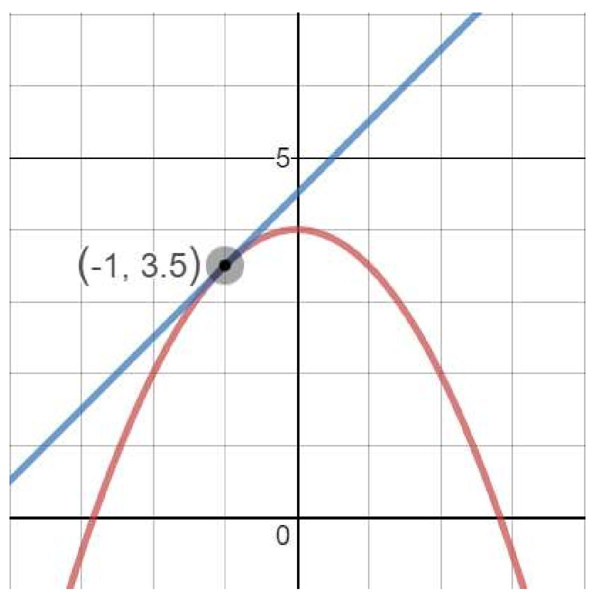
<u>Summary</u>

The derivative is a function whose outputs are equal to the slopes of a tangent line on a function. Using the geometric idea of the slope of a tangent line, we can select several points on a graph, draw a tangent line, and measure its slope. The resulting slope values may be graphed to obtain a graph of the derivative function.

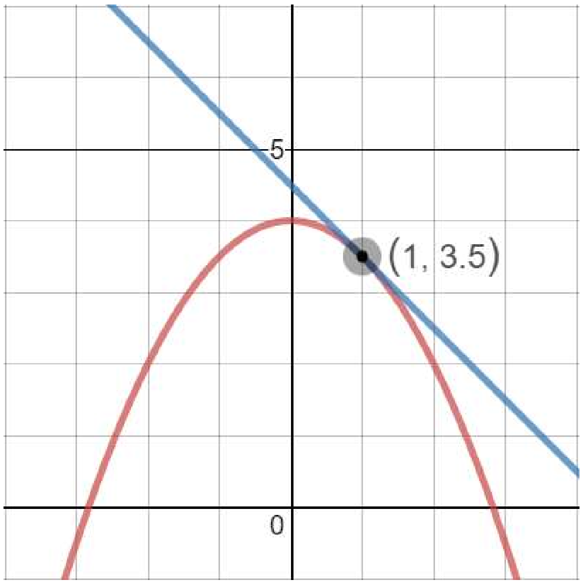
As an example, a parabola is graphed in red below with a tangent line in blue at x = -2, -1, 1, 2.



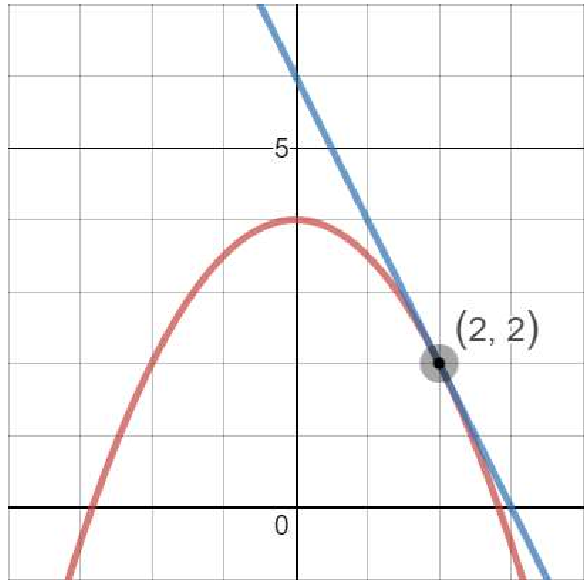
Slope of the tangent line at $x = -2$ is 2.



Slope of the tangent line at $x = -1$ is 1.



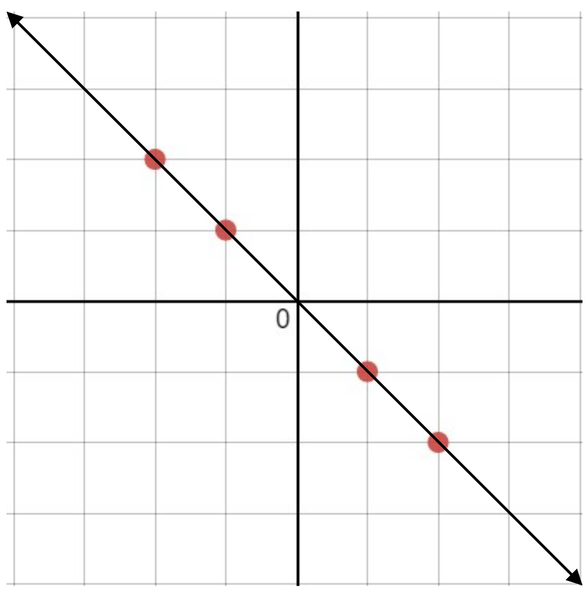
Slope of the tangent line at $x = 1$ is -1.



Slope of the tangent line at $x = 2$ is -2.

Let's put these values into a table and graph them:

x	$f'(x)$
-2	2
-1	1
1	-1
2	-2

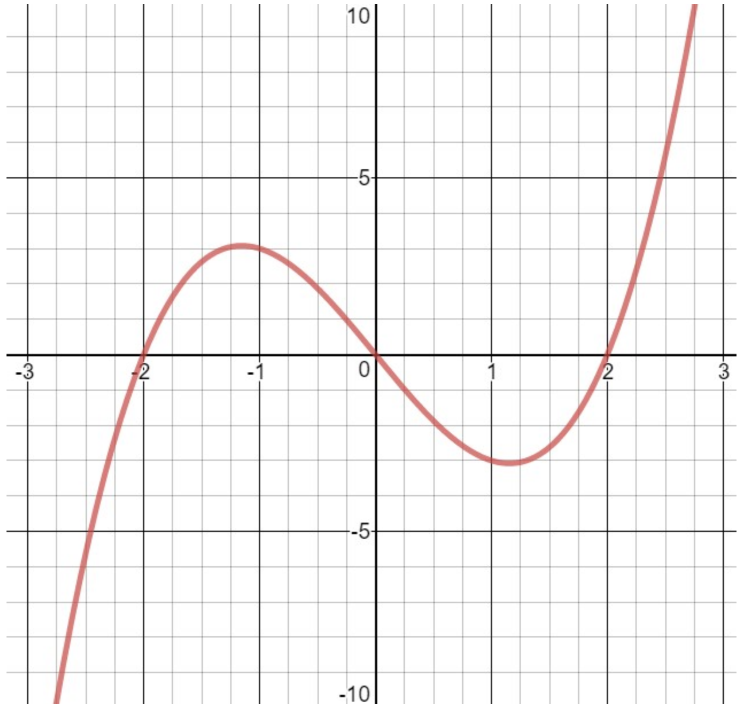


Clearly there is a pattern to these points. The derivative of the parabola is the linear function shown above

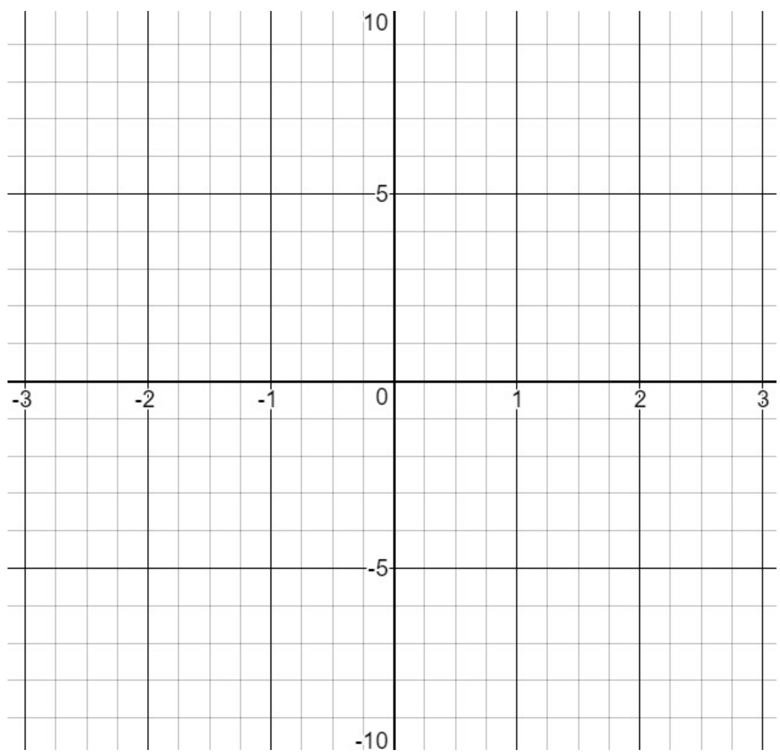
<u>Notes</u>

Practice

1. For the graph below, fill out the table with derivative values and graph them in the adjacent graph.



x	$f'(x)$



Question 2 – How do you calculate the derivative of a function from the definition?

Key Terms

Derivative	Difference Quotient
Function	

Summary

The derivative function is defined by a difference quotient like the instantaneous rate or the derivative at a point:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Note that the difference quotient contains x instead of a. This is fortunate since it allows us to use the same algebraic techniques from Chapter 10 to evaluate the limits.

Notes

Guided Example

Use the definition of the derivative to find the derivative of

$$f(x) = 3x^2 + 4x + 1$$

Solution To evaluate the limit in the definition,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \ ,$$

we need to evaluate $f(x+h)$ in the numerator. Replacing x with $x+h$ in the function gives

$$\begin{aligned} f(x+h) &= 3(x+h)^2 + 4(x+h) + 1 \\ &= 3x^2 + 6xh + 3h^2 + 4x + 4h + 1 \end{aligned}$$

Put this into the difference quotient along with $f(x)$:

$$\lim_{h \to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 + \cancel{4x} + 4h + \cancel{1} - \cancel{3x^2} - \cancel{4x} - \cancel{1}}{h}$$

$$\lim_{h \to 0} \frac{6xh + 3h^2 + 4h}{h}$$

$$\lim_{h \to 0} \frac{\cancel{h}(6x + 3h + 4)}{\cancel{h}}$$

$$\lim_{h \to 0} \ 6x + 3h + 4$$

$$3x + 4$$

So $f'(x) = 3x + 4$. Note that in the final limit, only h is approaching 0 so the derivative contains an x in it.

Practice

1. Use the definition of the derivative to find the derivative of

$$f(x) = -x^2 + 2x - 6$$

Guided Example	Practice
Use the definition of the derivative to find the derivative of $$g(x)=\frac{1}{x}$$ **Solution** Since the function is named $g(x)$, we need to adapt the definition to account for the different name: $$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$ As in the previous guided example, we need to find $g(x+h)$: $$g(x+h)=\frac{1}{x+h}$$ Put this into the difference quotient along with $g(x)$: $$g'(x)=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$ $$=\lim_{h\to 0}\frac{\frac{x}{x(x+h)}-\frac{x+h}{x(x+h)}}{h}$$ $$=\lim_{h\to 0}\frac{\frac{\cancel{x}-\cancel{x}-h}{x(x+h)}}{h}$$ $$=\lim_{h\to 0}\frac{-\cancel{h}}{x(x+h)}\cdot\frac{1}{\cancel{h}}$$ $$=\lim_{h\to 0}\frac{-1}{x(x+h)}$$ $$=\frac{-1}{x^2}$$ The derivative is $g'(x)=\frac{-1}{x^2}$	2. Use the definition of the derivative to find the derivative of $$g(x)=\frac{1}{2x}$$ Solution

Question 3 – What are the derivatives of some basic functions (linear, polynomial, power, exponential, and logarithmic)?

Key Terms

Derivative | Polynomial function

Exponential function | Logarithm function

Summary

Calculating the derivative from the definition is tedious and time consuming. Luckily, formulas exist for calculating derivatives of basic functions like polynomial, exponential, and logarithmic functions.

$\frac{d}{dx}[c]=0$	$\frac{d}{dx}[ax+b]=a$	$\frac{d}{dx}\left[x^n\right]=nx^{n-1}$
$\frac{d}{dx}[a\,f(x)]=a\ f'(x)$	$\frac{d}{dx}[f(x)\pm g(x)]=f'(x)\pm g'(x)$	$\frac{d}{dx}\left[e^x\right]=e^x$
$\frac{d}{dx}\left[a^x\right]=(\ln a)a^x$	$\frac{d}{dx}\left[\log_a(x)\right]=\frac{1}{\ln(a)}\cdot\frac{1}{x}$	$\frac{d}{dx}\left[\ln(x)\right]=\frac{1}{x}$

Notes

Guided Example	Practice
Find the derivative of $f(x)=4x^2-5x+7$ **Solution** Utilize the rules and especially the power rule for derivatives to give $f'(x)=\frac{d}{dx}\left[4x^2\right]-\frac{d}{dx}[5x]+\frac{d}{dx}[7]$ $=4\frac{d}{dx}\left[x^2\right]-5\frac{d}{dx}[x]+\frac{d}{dx}[7]$ $=4\cdot 2x-5\cdot 1+0$ $=8x-5$	1. Find the derivative of $f(x)=5x^4-6x^3+1$

Guided Example	Practice
If $y=\frac{2}{x^2}+\pi^3$, find $\frac{dy}{dx}$. **Solution** Start by rewriting the fraction with negative exponents: $y=2x^{-2}+\pi^3$ Now take the derivative of each term: $\frac{dy}{dx}=\frac{d}{dx}\left[2x^{-2}\right]+\frac{d}{dx}\left[\pi^3\right]$ $=2\frac{d}{dx}\left[x^{-2}\right]+\frac{d}{dx}\left[\pi^3\right]$ $=2\cdot -2x^{-3}+0$ $=-4x^{-3}$ The second term is zero since it is constant with respect to the variable *x*. Only powers on the same variables change with the power rule.	2. Find $D_t\left[\frac{1}{2t}+e^2\right]$

Guided Example | Practice

Guided Example	Practice
Find the derivative of $g(t)=\sqrt{t}-t^{-1/2}$ **Solution** Rewrite the function so that all terms are written with powers. Using fractional powers in place of the root we get $$g(t)=t^{1/2}-t^{-1/2}$$ Now utilize the power rule on each term to give $$g'(t)=\frac{d}{dt}\left[t^{1/2}\right]-\frac{d}{dt}\left[t^{-1/2}\right]$$ $$=\frac{1}{2}t^{-1/2}-\left(-\frac{1}{2}t^{-3/2}\right)$$ $$=\frac{1}{2}t^{-1/2}+\frac{1}{2}t^{-3/2}$$	3. Find the derivative of $P(q)=\dfrac{\sqrt[4]{q}+q+1}{q}$

Guided Example | Practice

Guided Example	Practice
Find $D_x\left[3\cdot 2^x\right]$ **Solution** Apply the derivative rule for exponentials to $$D_x\left[3\cdot 2^x\right]=3\,D_x\left[2^x\right]$$ $$=3\cdot\left(\ln 2\right)2^x$$	4. If $y=6\,e^x$, find $\dfrac{dy}{dx}$.

Guided Example	Practice
Find the derivative of $h(x)=\ln(x)+5x^3+10$ **Solution** Take the derivative of each term: $h'(x)=\frac{d}{dx}\left[\ln(x)\right]+\frac{d}{dx}\left[5x^3\right]+\frac{d}{dx}\left[10\right]$ $=\frac{d}{dx}\left[\ln(x)\right]+5\frac{d}{dx}\left[x^3\right]+\frac{d}{dx}\left[10\right]$ $=\frac{1}{x}+5\cdot 3x^2+0$ $=\frac{1}{x}+15x^2$	5. Find the derivative of $h(x)=\log_3(x)-e^6+1$

Section 11.5 Economic Applications of the Derivative

Question 1 – What does the term marginal mean?

Question 2 – How are derivatives used to compute elasticity?

Question 1 – What does the term marginal mean?

Key Terms

Marginal	Secant line
Tangent line	Domain

Summary

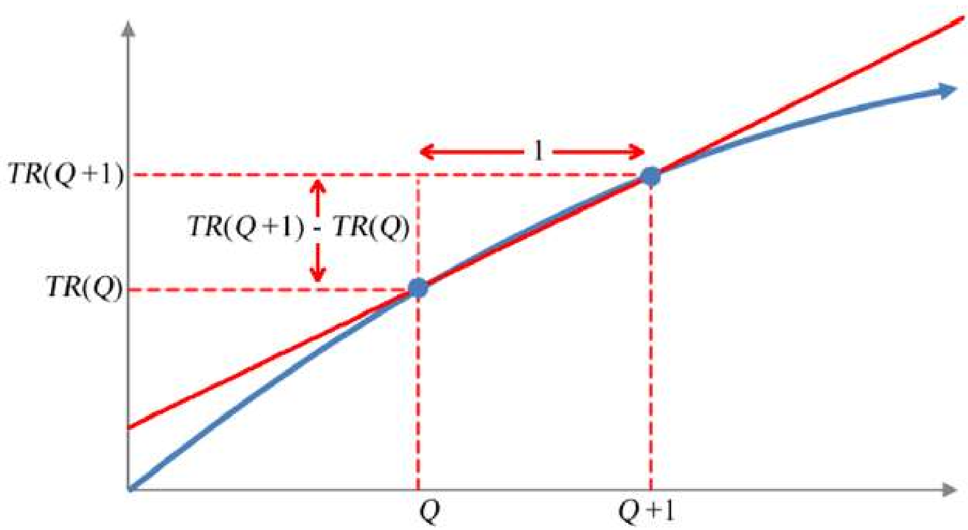
Marginal functions indicate how a quantity changes when production is increased by one unit. If total revenue at a production level Q is $TR(Q)$ and the total revenue at a production level of one higher is $TR(Q+1)$, then the marginal revenue is

$$TR(Q+1) - TR(Q)$$

Similarly, if the total cost at a production level Q is $TC(Q)$ and the total cost at a production level of one higher is $TC(Q+1)$, then the marginal cost is

$$TC(Q+1) - TC(Q)$$

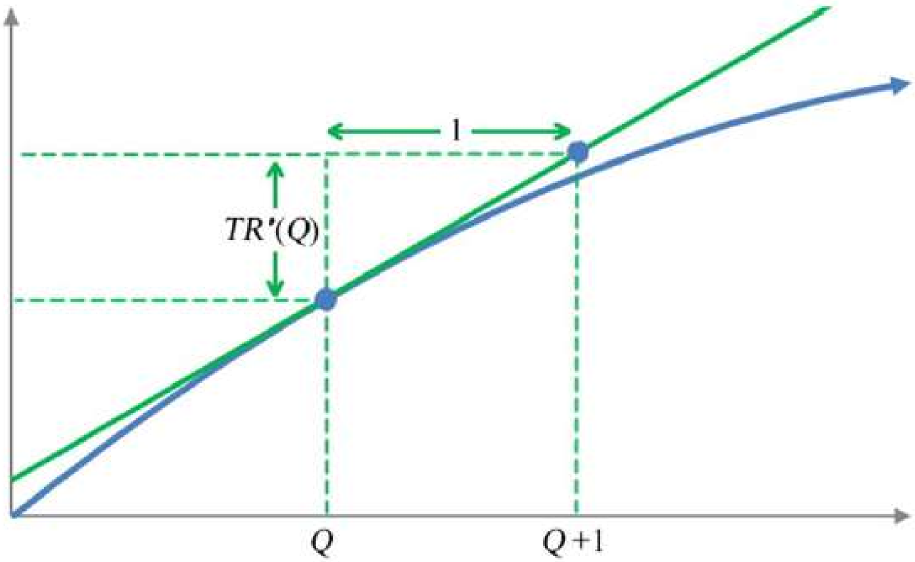
In fact, we can define a marginal function in a similar way for other functions like profit. As discussed in the textbook, these quantities can be viewed graphically as the slope of a secant line.



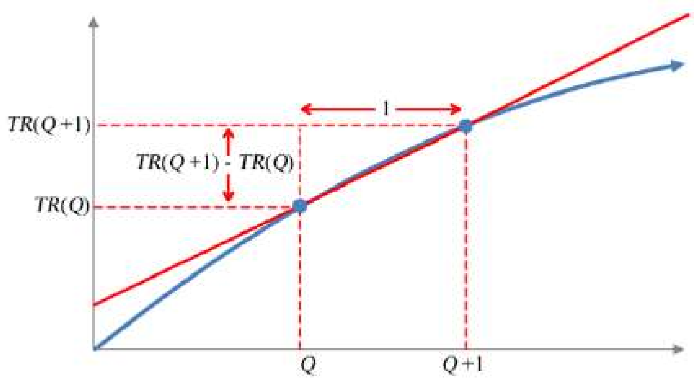
The slope of the secant line in this context is

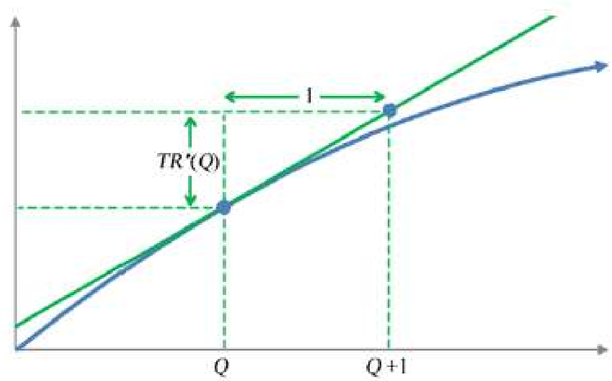
$$\text{Slope of Secant Line} = \frac{TR(Q+1) - TR(Q)}{1}$$

Now let's look at a very similar picture with a tangent line on it.



We know that the derivative $TR'(Q)$ gives the slope of the tangent line in the graph above. Now let's compare the two graphs.





The slope of the secant line (red line) on the left and the slope of the tangent line (green line) on the right are approximately the same. Because of this, we often approximate marginal revenue with the derivative,

$$\begin{array}{c}\text{Marginal Revenue at}\\ \text{a Produciton Level Q}\end{array} = TR(Q+1) - TR(Q) \approx TR'(Q)$$

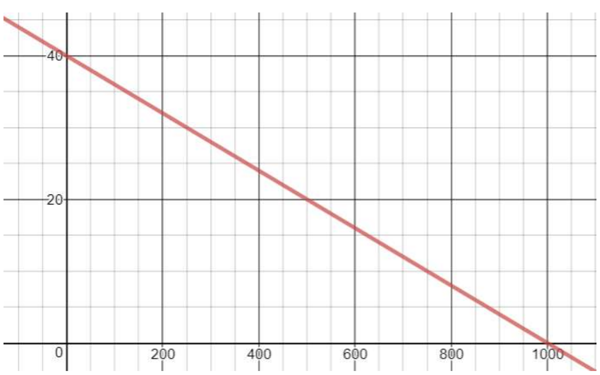
Keep in mind that this relationship is approximate since the slope of the secant line is not exactly the same of the slope of the tangent line. We can write the marginal cost in a similar manner,

$$\begin{array}{c}\text{Marginal Cost at}\\ \text{a Produciton Level Q}\end{array} = TC(Q+1) - TC(Q) \approx TC'(Q)$$

Notes

Guided Example	Practice
Suppose the demand function is given by $D(Q) = -0.05Q + 100$ dollars per unit where Q is the number of units demanded by consumers. a. Find and interpret the marginal total revenue at Q = 700 units. **Solution** To find the marginal total revenue, we first need to find the total revenue. This is done by multiplying the demand function by the quantity, $$\begin{aligned} TR(Q) &= Q(-0.05Q + 100) \\ &= -0.05Q^2 + 100Q \end{aligned}$$ Since the demand was in dollars per unit and the quantity is in units, the total revenue must be in dollars.To find the marginal total revenue, take the derivative, $$TR'(Q) = -0.10Q + 100$$	1. Suppose the demand function is given by $D(Q) = -0.01Q + 80$ dollars per unit where Q is the number of units demanded by consumers. a. Find and interpret the marginal total revenue at Q = 5000 units.

Now substitute Q = 700 into the marginal total revenue to give $$TR'(700) = -.10(700) + 100 = 30$$ In dollars per unit. This means that the 701st unit sold will increase revenue by 30 dollars.	
b. If the cost function is given by $TC(Q) = 9Q + 5650$ dollars, find and interpret the marginal total profit at Q = 700. **Solution** Start by finding the total profit. Subtract the total cost from the total revenue to give the total profit, $$TP(Q) = -0.05Q^2 + 100Q - (9Q + 5650)$$ $$= -0.05Q^2 + 91Q - 5650$$ The marginal total profit function is $$TP'(Q) = -0.10Q + 91$$ And the marginal total profit at 700 units is $$TP'(700) = -0.10(700) + 91 = 21$$ in dollars per unit. This means the 701st unit produced and sold will increase profit by \$21.	b. If the cost function is given by $TC(Q) = 15Q + 50{,}000$ dollars, find and interpret the marginal total profit at Q = 5000.

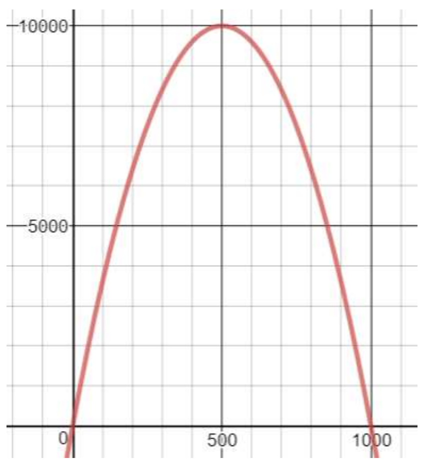
Guided Example	Practice
The price (in dollars) and the demand x for a particular product are related by the demand equation $x = 1000 - 25p$. a. Express the price p in term of the demand x, and find the domain of this function. **Solution** As it is written in the problem statement, the demand function is given as a function of p. To write it as a function of x, we need to solve the demand function for p: $x = 1000 - 25p$ $x - 1000 = -25p$ $\frac{x-1000}{-25} = p$ $-\frac{1}{25}x + 40 = p$ The domain is the reasonable values for x in this function. Let's graph the function to see what those values are.  Since x represent the number of products, it cannot be negative. Similarly, the price must also be nonnegative. This means that the domain begins at the vertical axis and extends until the horizontal intercept at x = 1000. So, the domain of the function is $0 \le x \le 1000$.	2. The price (in dollars) and the demand x for a particular product are related by the demand equation $x = 2000 - 20p$. a. Express the price p in term of the demand x, and find the domain of this function.

b. Find the revenue $R(x)$ from the sale of x products. What is the domain of the revenue R?

Solution The revenue function is calculated by multiplying the demand as a function of x by the number of products x:

$$R(x) = \left(-\tfrac{1}{25}x + 40\right)x$$
$$= -\tfrac{1}{25}x^2 + 40x$$

The graph of the revenue function helps us to see the domain of $R(x)$.



To make sure the number of products x and the revenue R(x) is nonnegative, the x values must be from 0 to 1000. This makes the domain $0 \le x \le 1000$.

b. Find the revenue $R(x)$ from the sale of x products. What is the domain of the revenue R?

c. Find and interpret the marginal revenue at a production level of 600. **Solution** The marginal revenue is the derivative of the revenue function, $$R'(x) = -\tfrac{2}{25}x + 40$$ At a production level of 600, the marginal revenue is $$R'(600) = -\tfrac{2}{25}(600) + 40 = -8$$ or -8 dollars per product. This means that the 601^{st} product sold will reduce revenue by \$8.	c. Find and interpret the marginal revenue at a production level of 1200.

Question 2 – How are derivatives used to compute elasticity?

Key Terms

Percent change Elasticity

Summary

Elasticity expresses how to quantities change with respect to each other with percent. To begin thinking in terms of percent, we need to identify what is mean by percent change. If a variable X changes from X to ΔX, then the percent change in X is defined to be

$$\text{Percent change in X} = \frac{\Delta X}{X}$$

Elasticity is defined in terms of percent change as

$$\text{Elasticity of Y with respect to X} = \frac{\text{Percent change in Y}}{\text{Percent change in X}}$$

In business and economics, we are often interested in the price elasticity of demand. Using the definition above, we would define this as

$$\text{Elasticity of Q with respect to P} = \frac{\text{Percent change in Q}}{\text{Percent change in P}}$$

This definition helps us to understand how changes in price affect the corresponding change in demand by consumers.

> If $E < -1$, then **demand is elastic** and a percent increase in price yields a larger percent decrease in demand.
>
> If $-1 < E < 0$, then **demand is inelastic** and a percent increase in price yields a smaller percent decrease in demand.
>
> If $E = -1$, then the **demand is unit-elastic** and a percent increase in price yields the same percent decrease in demand.

This the price elasticity of demand can be approximated with derivatives as

$$E \approx \frac{P}{Q}\frac{dQ}{dP}$$

To apply this approximation, we need to have a demand function as a function written as a function of P ($Q = D(P)$). If the function is not solved for Q, you will need to solve it for Q before taking the derivative $\frac{dQ}{dP}$.

Be aware that many business and finance textbooks will define elasticity as a positive number and use absolute value,

$$E \approx \left| \frac{P}{Q} \frac{dQ}{dP} \right|$$

This definition hides the fact that an increase in price leads to a <u>decrease</u> in quantity and so is not used in this course.

Revenue is maximized when the elasticity is equal to -1.

<u>Notes</u>

<u>Guided Example</u>	<u>Practice</u>
Suppose the quantity demanded by consumers in units is given by $Q = 5000 - 5P$ where P is the unit price in dollars. a. Find the price elasticity of demand with respect to price when $P = 200$. **Solution** The price elasticity of demand is $E \approx \frac{P}{Q}\frac{dQ}{dP}$ To use this approximation, we need to evaluate the derivative, $\frac{dQ}{dP} = -5$ Now substitute the demand equation and derivative into the elasticity approximation: $E \approx \frac{P}{5000-5P} \cdot -5 = \frac{-5P}{5000-5P}$ Now substitute $P = 200$, $E \approx \frac{-5(200)}{5000-5(200)} = -.25$ This means that a price increase of 1% corresponds to a decrease in quantity of .25% when the price is \$200.	1. Suppose the quantity demanded by consumers in units is given by $Q = 100 - \frac{P}{2}$ where P is the unit price in dollars. a. Find the price elasticity of demand with respect to price when $P = 110$.

b. Find the quantity at which revenue is maximized. **Solution** The revenue is maximized when the price elasticity of demand is equal to -1. To find the point at which revenue is maximized, set the elasticity equal to -1 and solve for *P*: $\frac{-5P}{5000-5P} = -1$ $-5P = -5000 + 5P$ Multiply both sides by 5000 – 5*P* $-10P = -5000$ $P = 500$ This is the price at which revenue is maximized. To find the quantity, substitute the price into the demand equation, $Q = 5000 - 5(500) = 2500$	b. Find the quantity at which revenue is maximized.

Guided Example

Suppose the quantity demanded by consumers in units is given by $Q = 500 - 0.1P^2$ where P is the unit price in dollars.

a. Find the elasticity of demand with respect to price when $P = 100$.

Solution The price elasticity of demand is

$$E \approx \frac{P}{Q}\frac{dQ}{dP}$$

To use this approximation, we need to evaluate the derivative,

$$\frac{dQ}{dP} = -0.2P$$

Now substitute the demand equation and derivative into the elasticity approximation:

$$\begin{aligned} E &\approx \frac{P}{500 - 0.1P^2} \cdot -0.2P \\ &= \frac{-0.2P^2}{500 - 0.1P^2} \end{aligned}$$

Now substitute $P = 200$,

$$E \approx \frac{-0.2(100)^2}{500 - 0.1(100)^2} = 4$$

This means that a price increase of 1% corresponds to a increase in quantity of 4% when the price is \$100.

Practice

3. Suppose the quantity demanded by consumers in units is given by $Q = 600 - 0.2P^2$ where P is the unit price in dollars.

   a. Find the elasticity of demand with respect to price when $P = 20$.

b. Find the quantity at which revenue is maximized. **Solution** The revenue is maximized when the price elasticity of demand is equal to -1. To find the point at which revenue is maximized, set the elasticity equal to -1 and solve for *P*: $\frac{-0.2P^2}{500-0.1P^2} = -1$ $-0.2P^2 = -500 + 0.1P^2$ Multiply both sides by $500 - 0.1P^2$ $-0.3P^2 = -500$ $P^2 = \frac{500}{0.3}$ $P = \sqrt{\frac{500}{0.3}} \approx 40.82$ To find the quantity, substitute this price into the demand equation, $Q = 500 - 0.1\left(\sqrt{\frac{500}{0.3}}\right) \approx 483.33$	b. Find the quantity at which revenue is maximized.

Section 11.6 Derivatives of Products and Quotients

Question 1 – How do you find the derivative of a product of two functions?

Question 2 – How do you find the derivative of a quotient of two functions?

Question 3 - What is an average cost function?

Question 1 – How do you find the derivative of a product of two functions?

Key Terms

Product of two functions

Summary

Products are frequently encountered in business and finance. Products are where two functions are multiplied together. If the two functions multiplied together are called $u(x)$ and $v(x)$, we can take the derivative of the product using the Product Rule for Derivatives:

$$\frac{d}{dx}\left[u(x)v(x)\right] = v(x)u'(x) + u(x)v'(x)$$

This can be more easily memorized as

$$\frac{d}{dx}\left[uv\right] = vu' + uv'$$

Notes

Guided Example | Practice

Guided Example	Practice
Find the derivative of $y = xe^x$. **Solution** This function may be thought of as a product with $u = x$ and $v = e^x$. To apply the product rule, we need to take the derivative of each of these factors: $u = x \qquad u' = 1$ $v = e^x \qquad v' = e^x$ Now apply the product rule, $\frac{d}{dx}[uv] = vu' + uv'$ to give the derivative, $\frac{dy}{dx} = \underbrace{e^x}_{v} \cdot \underbrace{1}_{u'} + \underbrace{x}_{u} \cdot \underbrace{e^x}_{v'}$	1. Find the derivative of $y = (x^2 + 1)5^x$

Guided Example | Practice

Guided Example	Practice
Find $D_x\left[(x^2 + 7x - 5)\ln(x)\right]$. **Solution** Start by identifying the two factors and their derivatives: $u = x^2 + 7x - 5 \qquad u' = 2x + 7$ $v = \ln(x) \qquad v' = \frac{1}{x}$ Now put these pieces into the product rule to yield $\frac{dy}{dx} = \ln(x) \cdot \left(2x + 7\right) + \left(x^2 + 7x - 5\right) \cdot \frac{1}{x}$	2. Find $\frac{d}{dx}\left[(2x^3 - 5x + 1)\log(x)\right]$

Question 2 – How do you find the derivative of a quotient of two functions?

Key Terms

Quotient of two functions

Summary

When two functions are divided, the result is called a quotient. If the two pieces of the quotient are called $u(x)$ and $v(x)$, the we can take the derivative of the quotient with the Quotient Rule for derivatives:

$$\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right]=\frac{v(x)u'(x)-u(x)v'(x)}{\left(v(x)\right)^2}$$

This rule is more easily memorized in the form

$$\frac{d}{dx}\left[\frac{u}{v}\right]=\frac{vu'-uv'}{v^2}$$

Notes

Guided Example	Practice
Find $D_t\left[\frac{\log(t)}{t}\right]$ **Solution** Identify u and v so that we can apply the quotient rule, $$\frac{d}{dt}\left[\frac{u}{v}\right]=\frac{vu'-uv'}{v^2}$$ In this case, $$u=\log(t) \qquad u'=\frac{1}{\ln(10)}\cdot\frac{1}{t}$$ $$v=t \qquad v'=1$$ Now put these factors into the quotient rule, $$\frac{dy}{dt}=\frac{t\cdot\frac{1}{\ln(10)}\cdot\frac{1}{t}-\log(t)\cdot 1}{t^2}$$ $$=\frac{\frac{1}{\ln(10)}-\log(t)}{t^2}$$	1. Find the $\frac{dy}{dt}$ if $y=\frac{\ln(t)}{t}$.

Guided Example	Practice
If $f(x)=\dfrac{x+1}{x^2+4}$, find $f'(x)$. **Solution** For this function, $u=x+1 \qquad u'=1$ $v=x^2+4 \qquad v'=2x$ Put these factors into the quotient rule to give $f'(x)=\dfrac{\left(x^2+4\right)\cdot 1-(x+1)\cdot 2x}{\left(x^2+4\right)^2}$ $=\dfrac{x^2+4-2x^2-2x}{\left(x^2+4\right)^2}$ $=\dfrac{-x^2-2x+4}{\left(x^2+4\right)^2}$	2. If $g(t)=\dfrac{t^2-5t+10}{t+2}$, find $g'(t)$.

Question 3 – What is an average cost function?

Key Terms

Average Cost

Summary

If the total cost of producing Q items is $TC(Q)$, then the average cost function is found by dividing the total cost by the number of units Q,

$$\overline{TC}(Q) = \frac{TC(Q)}{Q}$$

The bar over the TC indicates average total cost. In many cases you will see the cost defined at x units as $C(x)$. In this case, you would write the average cost as

$$\overline{C}(x) = \frac{C(x)}{x}$$

The marginal average cost is simply the derivative of the average cost.

Notes

Guided Example	Practice
Suppose the total cost of producing Q items is given by $TC(Q) = 5Q + 6000$ thousand dollars. a. Find the average cost function $\overline{TC}(Q)$. **Solution** To find the average cost function, divide the total cost by the quantity, $$\overline{TC}(Q) = \frac{5Q + 6000}{Q}$$ Since the total cost is in thousands of dollars, the average cost is in thousands of dollars per item.	1. Suppose the total cost of producing x items is given by $C(x) = 1.5x + 300$ thousand dollars. a. Find the average cost function $\overline{C}(x)$.
b. Find and interpret $\overline{TC}(200)$. **Solution** Substitute $Q = 200$ into the average cost to yield $$\overline{TC}(200) = \frac{5(200) + 6000}{200} = 35$$ The average cost is 35 thousand dollars per item which means that each item costs $35,000 to make.	b. Find and interpret $\overline{C}(100)$.
c. Find the marginal average cost function. Solution To find the marginal average cost function, we need to take the derivative of the average cost function $\overline{TC}(Q) = \frac{5Q + 6000}{Q}$. Since the function is a quotient, let $$\begin{aligned} u &= 5Q + 6000 \qquad & u' &= 5 \\ v &= Q \qquad & v' &= 1 \end{aligned}$$ and put these factors into the quotient rule,	c. Find the marginal average cost function.

$$\overline{TC}'(Q) = \frac{Q \cdot 5 - (5Q + 6000) \cdot 1}{Q^2}$$ $$= \frac{-6000}{Q^2}$$ The units on this function are thousands of dollars per unit per unit. d. Find and interpret the marginal average cost function when 200 items are made. **Solution** Substitute $Q = 200$ into the function from part c. This gives $$\overline{TC}'(200) = \frac{-6000}{200^2} = -0.15$$ This means that when production in increased by 1 item, the average cost will decrease by 0.15 thousand dollars per unit.	d. Find and interpret the marginal average cost function when 100 items are made.

Section 11.7 The Chain Rule

Question 1 – What is a composition of two functions?

Question 2 – How do you write a function as a composition of two functions?

Question 3 - How do you apply the chain rule to take a derivative?

Question 4 - How do you combine derivative rules to take more complicated derivatives?

Question 1 – What is a composition of two functions?

Key Terms

Composition

Summary

When we compose two functions, we substitute one function in place of the variable in the other function. If the two functions are called f (x) and $g(x)$, then the composition is named $f \circ g$ or g $g \circ f$. The symbol ° indicates composition and would be read as "circle" or "composed with". The difference between $f \circ g$ and $g \circ f$ has to do with which function is substituted into which function. By definition, $f \circ g$ is obtained by substituting $g(x)$ into $f(x)$, $f\left(g(x)\right)$. The composition $g \circ f$ is obtained by substituting $f(x)$ into $g(x)$, $g\left(f(x)\right)$.

Notes

Guided Example	Practice
Suppose $f(x)=\frac{1}{2}x+7$ and $g(x)=5x+6$. a. Find $f(g(x))$. **Solution** Start by replacing $g(x)$ with its formula. Then place that function into $f(x)$: $f\left(g(x)\right)=f\left(5x+6\right)$ $=\frac{1}{2}\left(5x+6\right)+7$ b. Find $g(f(x))$. **Solution** Start by replacing $f(x)$ with its formula. Then place that function into $g(x)$: $g\left(f(x)\right)=g\left(\frac{1}{2}x+7\right)$ $=5\left(\frac{1}{2}x+7\right)+6$	1. Suppose $f(x)=\frac{3}{4}x+1$ and $g(x)=2x-4$. a. Find $f(g(x))$. b. Find $g(f(x))$.

Guided Example	Practice
Suppose $f(x)=\dfrac{1}{x}$ and $g(x)=2x^2-x+2$. a. Find $f(g(x))$. **Solution** Start by replacing $g(x)$ with its formula. Then place that function into $f(x)$: $f\left(g(x)\right)=f\left(2x^2-x+2\right)$ $=\dfrac{1}{2x^2-x+2}$ b. Find $g(f(x))$. **Solution** Start by replacing $f(x)$ with its formula. Then place that function into $g(x)$: $g\left(f(x)\right)=g\left(\frac{1}{x}\right)$ $=2\left(\frac{1}{x}\right)^2-\left(\frac{1}{x}\right)+2$	2. Suppose $f(x)=\dfrac{6}{x}$ and $g(x)=x^3-x^2-5$. a. Find $f(g(x))$. b. Find $g(f(x))$.

Guided Example	Practice
Suppose $f(x)=\sqrt{x-2}$ and $g(x)=\frac{2}{x}$. a. Find $f(g(x))$. **Solution** Start by replacing $g(x)$ with its formula. Then place that function into $f(x)$: $f\left(g(x)\right)=f\left(\frac{2}{x}\right)$ $=\sqrt{\frac{2}{x}-2}$ b. Find $g(f(x))$. **Solution** Start by replacing $f(x)$ with its formula. Then place that function into $g(x)$: $g\left(f(x)\right)=g\left(\sqrt{x-2}\right)$ $=\frac{2}{\sqrt{x-2}}$	3. Suppose $f(x)=\sqrt[3]{x+1}$ and $g(x)=\frac{1}{x}$. a. Find $f(g(x))$. b. Find $g(f(x))$.

Guided Example	Practice
Suppose the demand for a certain product is given by $D(p)=\frac{1}{p+1}$, where p is in dollars. If the price, in terms of cost c, is expressed as $p(c)=5c-10$, find the demand function in terms of cost. **Solution** To find the demand function as a function of cost, we need to compose the demand function with the price function. This allows us to combine the price function (which takes in cost and outputs price) with the demand function (which takes in price and outputs demand). $D\left(p(c)\right)=D(5c-10)$ $=\frac{1}{5c-10+1}$	4. Suppose the demand for a certain product is given by $D(p)=\frac{-p^2}{50}+125$, where p is in dollars. If the price, in terms of cost c, is expressed as $p(c)=2c+15$, find the demand function in terms of cost.

Question 2 – How do you write a function as a composition of two functions?

Key Terms

Decomposition

Summary

When function is written as a composition of two functions, we say the function has been decomposed. This process is the reverse of composing two functions. Suppose we have a function $h(x)=\sqrt{x^2+1}$. To see how we can decompose this into $f(x)$ and $g(x)$ so that $f\left(g(x)\right)$, let's examine what the function does. If we put an input into $h(x)$, the input is

1. Squared
2. Add 1
3. Square root

When this function is decomposed, one of the functions needs to perform two of these steps and the other need to perform the other step.

$g(x)=x^2$	1. Squared	$g(x)=x^2+1$
	2. Add 1	
$f(x)=\sqrt{x+1}$	3. Square root	$f(x)=\sqrt{x}$

Either decomposition will lead to the same function when composed as $f\left(g(x)\right)$.

Notes

Guided Example

Practice

Guided Example	Practice
Write the function below as a decomposition of two functions in the form $f(g(x))$. $$y=\left(x^2+1\right)^6$$ **Solution** If we list out the steps that this function performs on an input, we get 1. Square 2. Add 1 3. Raise to the 6^{th} power We can decompose these steps as $g(x)=x^2$ and $f(x)=\left(x+1\right)^6$ or as $g(x)=x^2+1$ and $f(x)=x^6$. Either decomposition results in the same function when composed.	1. Write the function below as a decomposition of two functions in the form $f(g(x))$. $$y=\left(x^3+x\right)^5$$

Guided Example

Practice

Guided Example	Practice
Write the function below as a decomposition of two functions in the form $f(g(x))$. $$y=\sqrt{x^2-2x+5}$$ **Solution** This function is more complicated since there is more than one place to put the input. However, both of those inputs are in a polynomial. So, let $g(x)=x^2-2x+5$. To get the proper composition, the other function must be $f(x)=\sqrt{x}$.	2. Write the function below as a decomposition of two functions in the form $f(g(x))$. $$y=\sqrt[3]{2x^2+x-15}$$

Question 3 – How do you apply the chain rule to take a derivative?

Key Terms

Chain rule Differentiable

Summary

If a function may be written as a composition $f\left(g(x)\right)$, its derivative is computed using the Chain Rule. In the Chain Rule, we decompose the function into two differentiable functions $f(x)$ and $g(x)$. Then the derivative of the composition is

$$\frac{d}{dx}\left[f\left(g(x)\right)\right] = f'\left(g(x)\right) g'(x)$$

Notice that the chain rule results in a product. In the first factor, $g(x)$ is substituted into the derivative $f'(x)$. The second factor is the derivative $g'(x)$.

Notes

Guided Example | Practice

Guided Example	Practice
Find the derivative of the function, $$y=\left(x^3-2x+6\right)^5$$ **Solution** Start by decomposing the function into $$f(x)=x^5 \qquad g(x)=x^3-2x+6$$ To apply the chain rule, we need the derivatives of each of these pieces: $$f'(x)=5x^4 \qquad g'(x)=3x^2-2$$ Now put these pieces into the Chain Rule, $$\frac{dy}{dx}=\underbrace{5\left(x^3-2x+6\right)^4}_{f'(g(x))}\underbrace{\left(3x^2-2\right)}_{g'(x)}$$	1. Find the derivative of the function, $$y=\left(x^2+1\right)^{10}$$

Guided Example	Practice
Find the derivative of the function, $$y=-4\left(2x^3+3x^2\right)^{-4}$$ **Solution** Start by decomposing the function into $$f(x)=-4x^{-4} \qquad g(x)=2x^3+3x^2$$ To apply the chain rule, we need the derivatives of each of these pieces: $$f'(x)=16x^{-5} \qquad g'(x)=6x^2+6x$$ Now put these pieces into the Chain Rule, $$\frac{dy}{dx}=\underbrace{16\left(2x^3+3x^2\right)^{-5}}_{f'(g(x))}\underbrace{\left(6x^2+6x\right)}_{g'(x)}$$	2. Find the derivative of the function, $$y=\left(x^2+10\right)^{-5}$$

Guided Example | Practice

Guided Example	Practice
Find the derivative of the function, $$y = 10(2t+7)^{2/3}$$ **Solution** Start by decomposing the function into $$f(t) = 10t^{2/3} \qquad g(t) = 2t - 7$$ To apply the chain rule, we need the derivatives of each of these pieces: $$f'(t) = \tfrac{20}{3}t^{-1/3} \qquad g'(t) = 2$$ Now put these pieces into the Chain Rule, $$\frac{dy}{dt} = \underbrace{\tfrac{20}{3}(2t-7)^{-1/3}}_{f'(g(t))}\underbrace{(2)}_{g'(t)}$$	3. Find the derivative of the function, $$y = 6(3t-5)^{1/3}$$

Guided Example | Practice

Guided Example	Practice
The cost for producing x units of a product is $$C(x) = \sqrt{2x^2 + 250}$$ where C is the cost in dollars. Find the marginal cost of producing 35 units and interpret the answer. **Solution** To find the marginal cost, we need to take the derivative of the cost function. This requires us to use the Chain Rule with $$f(x) = \sqrt{x} \qquad g(x) = 2x^2 + 250$$ To apply the chain rule, we need the derivatives of each of these pieces: $$f'(x) = \tfrac{1}{2}x^{-1/2} \qquad g'(x) = 4x$$ Now put these pieces into the Chain Rule,	4. The cost for producing x units of a product is $$C(x) = \sqrt[3]{5x+10}$$ where C is the cost in dollars. Find the marginal cost of producing 10 units and interpret the answer.

$$C'(x) = \underbrace{\tfrac{1}{2}\left(2x^2+250\right)^{-1/2}}_{f'(g(x))}\underbrace{(4x)}_{g'(x)}$$ $$= \frac{4x}{2\sqrt{2x^2+250}}$$ $$= \frac{2x}{\sqrt{2x^2+250}}$$ To find the marginal cost at 35 units, put $x = 35$ into the marginal cost function, $$C'(35) = \frac{2(35)}{\sqrt{2(35)^2+250}} \approx 1.35$$ A marginal cost of 1.35 dollars per unit means that producing the 36th unit will increase cost by \$1.35.	

Question 3 – How do you combine derivative rules to take more complicated derivatives?

Key Terms

Product Rule

Quotient Rule

Chain Rule

Summary

In more complicated derivatives, the Chain Rule may be combined with the Product Rule,

$$\frac{d}{dx}\left[uv\right] = vu' + uv'$$

or the Quotient Rule,

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$$

Separating the rules from the Chain Rule help you to process the rules individually so they are not confused.

Notes

Guided Example

Find the derivative of the function,

$$y = \left(x^2 + 4x\right)\left(5x + 6\right)^4$$

Solution Start by noticing that this function is a product with a composition in the second piece of the product. This tells us that we will start with the Product Rule and then apply the Chain Rule when we need the derivative of the second factor.

Starting with the Product Rule, we get

$$\frac{dy}{dx} = \left(5x+6\right)^4\left(2x+4\right) + \left(x^2+4x\right)\cdot\frac{d}{dx}\left[\left(5x+6\right)^4\right]$$

The Product Rule has been written out. Notice that we left one of the derivatives in symbolic form. Next, we will use the Chain Rule to carry out that derivative:

$$\frac{d}{dx}\left[\left(5x+6\right)^4\right] = 4\left(5x+6\right)^3\left(5\right)$$

Placing this in our Product Rule completes the derivative,

$$\frac{dy}{dx} = \left(5x+6\right)^4\left(2x+4\right) + \left(x^2+4x\right)\cdot 4\left(5x+6\right)^3\left(5\right)$$

Practice

1. Find the derivative of the function,

$$y = x\left(2x^2 - x\right)^3$$

Guided Example

Find the derivative of the function,

$$y = x\, e^{x^2+1}$$

Solution This is a Product Rule with a Chain Rule inside of one of the factors. Using the Product Rule, we get

$$\frac{dy}{dx} = e^{x^2+1}(1) + x \cdot \frac{d}{dx}\left[e^{x^2+1}\right]$$

Applying the Chain Rule to the derivative that still needs to be completed, we get

$$\frac{d}{dx}\left[e^{x^2+1}\right] = e^{x^2+1}(2x)$$

Combining this with the Product Rule above gives

$$\frac{dy}{dx} = e^{x^2+1}(1) + x \cdot e^{x^2+1}(2x)$$

Practice

2. Find the derivative of the function,

$$y = x^3 \ln(2x+1)$$

Guided Example

Find the derivative of the function,

$$y = \frac{2x+1}{\ln(4x+7)}$$

Solution This derivative requires us to use the Quotient Rule followed by the Chain Rule. Starting with the Quotient Rule, we get

$$\frac{dy}{dx} = \frac{\ln(4x+7)(2) - (2x+1)\frac{d}{dx}\left[\ln(4x+7)\right]}{\left(\ln(4x+7)\right)^2}$$

Now apply the Chain Rule to finish the remaining derivative:

$$\frac{d}{dx}\left[\ln(4x+7)\right] = \frac{1}{4x+7} \cdot 4$$

Put this derivative into the Quotient Rule to give

$$\frac{dy}{dx} = \frac{\ln(4x+7)(2) - (2x+1)\left(\frac{4}{4x+7}\right)}{\left(\ln(4x+7)\right)^2}$$

Practice

3. Find the derivative of the function,

$$y = \frac{\ln(2x+1)}{4x+7}$$

Chapter 11 Answers

Section 11.1

Question 1 1) a. 0.05 billion dollars per year, b. approximately -0.33 billion dollars per year, c. Capital expenditures rose over 2006 to 2008, but then decreased over 2008 to 2011.

2) a. approximately 6.97 billion dollars per year, b. 11.9 million connections per year, c. 0.585 thousand dollars per connection or 585 dollars per connection.

Question 2 1) 332.5 million dollars per year

Section 11.2

Question 1 1) a. approximately 0.0046 million barrels per employee, b. 0.005 million barrels per employee.

Question 2 1) -52 items per dollar

Section 11.3

Question 1 1) approximately 4

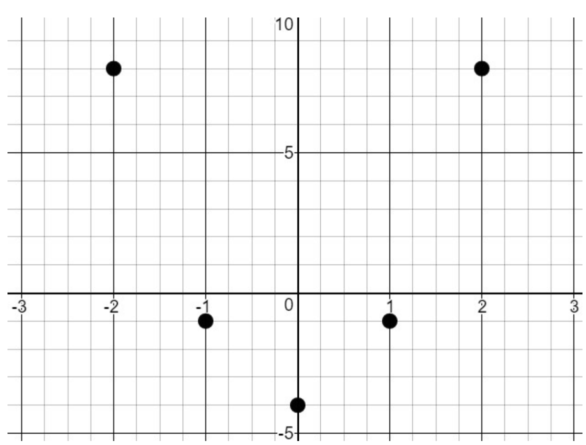
Question 2 1) 5

Question 3 1) a. $y = 2$, b. 2

Question 4 1) $P'(7) = 10$. We would say that the profit is changing by 10 tens of dollars per calculator or more simply 100 dollars per calculator. Another way to say this is that the 701st calculator increases profit by 100 dollars.

Section 11.4

Question 1 1)



Question 2 1) $-2x+2$, 2) $\frac{-1}{2x^2}$

Question 3 1) $20x^3-18x^2$, 2) $\frac{-1}{2t^2}$, 3) $-\frac{3}{4}q^{-7/4}-q^{-2}$, 4) $6e^x$, 5) $\frac{1}{\ln(3)}\frac{1}{x}$

Section 11.5

Question 1 1) a. -20, 5001st unit drops revenue by \$20, b. -35, 5001st unit drops profit by \$35

2) a. $p=-\frac{1}{20}x+100$, domain is $0\le x\le 2000$, b. $R(x)=-\frac{1}{20}x^2+100x$, domain is $0\le x\le 2000$, c. -20, 1201st unit decreases revenue by \$20

Question 2 1) a. $-\frac{110}{90}$, b. At a price of \$100, but a quantity of 50.

2) a. -0.308, b. At a price of \$31.62, but a quantity of 400.

Section 11.6

Question 1 1) $\frac{dy}{dx}=5^x(2x)+(x^2+1)\ln(5)5^x$

2) $\log(x)(6x^2-5)+(2x^3-5x+1)\frac{1}{\ln(10)}\frac{1}{x}$

Question 2 1) $\frac{dy}{dt}=\frac{t\left(\frac{1}{t}\right)-\ln(t)(1)}{t^2}$

2) $g'(t)=\frac{(t+2)(2t-5)-(t^2-5t+10)(1)}{(t+2)^2}$

Question 3 1) a. $\overline{C}(x)=\frac{1.5x+300}{x}\frac{\text{thousand \$}}{\text{unit}}$, b. $\overline{C}(100)=4.5$ thousand \$ per unit which means that each unit costs an average of \$4500 when 100 units are produced, c. $\overline{C}'(x)=-\frac{300}{x^2}$, d. $\overline{C}'(200)=-0.0075\frac{\frac{\text{thousand \$}}{\text{unit}}}{\text{unit}}$ so when the 201st item is produced, average cost decreases \$7.50 per unit.

Section 11.7

Question 1 1) a. $\frac{3}{4}(2x-4)+1$, b. $2\left(\frac{3}{4}x+1\right)-4$

2) a. $\dfrac{6}{x^3-x^2-5}$, b. $\left(\frac{6}{x}\right)^3-\left(\frac{6}{x}\right)^2-5$

3) a. $\sqrt[3]{\frac{1}{x}+1}$, b. $\dfrac{1}{\sqrt[3]{x+1}}$

4) $D\left(p(c)\right)=\dfrac{-(2c+15)^2}{50}+125$

Question 2 1) Many possible answers but most likely is $f(x)=x^5$ and $g(x)=x^3+5$.

2) Many possible answers but most likely is $f(x)=\sqrt[3]{x}$ and $g(x)=2x^2-x-15$.

Question 3 1) $\dfrac{dy}{dx}=10\left(x^2+1\right)^9\cdot 2x$

2) $\dfrac{dy}{dx}=-5\left(x^2+1\right)^{-6}\cdot 2x$

3) $\dfrac{dy}{dt}=6\cdot\frac{1}{3}(3t-5)^{-2/3}\cdot 3$

4) $C'(x)=\frac{1}{3}(5x+10)^{-2/3}\cdot 5$ so $C'(10)\approx 0.11$ dollars per unit. This means that the 11^{th} unit produced will increase cost by \$0.11.

Question 4 1) $\dfrac{dy}{dx}=\left(2x^2-x\right)^3\cdot 1+x\cdot 3\left(2x^2-x\right)^2(4x-1)$

2) $\dfrac{dy}{dx}=\ln(2x+1)\cdot 3x^2+x^3\cdot\dfrac{1}{2x+1}\cdot 2$

3) $\dfrac{dy}{dx}=\dfrac{(4x+7)\cdot\dfrac{1}{2x+1}\cdot 2-\ln(2x+1)\cdot 4}{(4x+7)^2}$