Section 14.1 The Substitution Method

Question 1 – How do we find the antiderivative of functions involving compositions?

Question 2 – How is the exact area under a function involving compositions computed?

Question 3 - How do you undo a rate with the Substitution Method?

Question 1 – How do we find the antiderivative of functions involving compositions?

Key Terms

Substitution method

Summary

To understand how the substitution works in computing antiderivatives, we need to first make sure we understand how the chain rule works. Let's start by taking the derivative of

$$y = \left(x^2 + 2x\right)^7$$

To apply the chain rule to this function, we identify the inside part, g(x), of the right side as

$$g(x) = x^2 + 2x$$

and the outside part, f(x), as

$$f(x) = x^7$$

This means that the function is being written as a composition in the form $f(g(x))$. The derivatives of these functions are

$$g(x) = x^2 + 2x \quad \rightarrow \quad g'(x) = 2x + 2$$

$$f(x) = x^7 \quad \rightarrow \quad f'(x) = 7x^6$$

This results in the derivative

$$\frac{dy}{dx} = \underbrace{7\left(x^2 + 2x\right)^6}_{f'(g(x))}\underbrace{\left(2x + 2\right)}_{g'(x)}$$

The corresponding antiderivative would be

$$\int 7\left(x^2 + 2x\right)^6 \left(2x + 2\right) dx = \left(x^2 + 2x\right)^7 + C$$

The derivative and antiderivative are opposite processes of each other:

$$\left(x^2+2z\right)^7 \quad \underset{\text{u substitution}}{\overset{\text{chain rule}}{\rightleftarrows}} \quad 7\left(x^2+2x\right)^6\left(2x+2\right)$$

The opposite process of the chain rule is called u substitution. In this antiderivative technique, the inside function g(x) is called u and is used to simplify the integrand. Let's look at how this is done. We'll find the antiderivative

$$\int 7\left(x^2+2x\right)^6\left(2x+2\right)dx$$

Identify the inside function as $u=x^2+2x$. The derivative of the inside function is $\frac{du}{dx}=2x+2$. Multiply each side by dx to give

$$\color{red}{dx}\cdot\frac{du}{dx}=\left(2x+2\right)\cdot\color{red}{dx}$$

$$du=\left(2x+2\right)dx$$

We can find u and du in the integrand:

$$\int 7\underbrace{\left(x^2+2x\right)}_{u}{}^6\underbrace{\left(2x+2\right)dx}_{du}=\int 7u^6\,du$$

Notice that this is simply the antiderivative of the outside function, $f'\left(u\right)$. We can evaluate this antiderivative with the power rule for antiderivatives,

$$\int 7u^6\,du=u^7+C$$

Since the original variable in the p[problem was x, we need to get back to that variable using $u=x^2+2x$. This means the antiderivative is $\left(x^2+2x\right)^7+C$ or

$$\int 7\left(x^2+2x\right)^6\left(2x+2\right)dx=\left(x^2+2x\right)^7+C$$

Now let's review this all put together:

$$\int 7\underbrace{\left(x^2+2x\right)}_{u}{}^{6}\underbrace{\left(2x+2\right)dx}_{du} = \int 7u^6\,du$$

$$= 7\int u^6\,du$$

$$= 7\cdot\tfrac{1}{7}u^7 + C$$

$$= u^7 + C$$

$$= \left(x^2+2x\right)^7 + C$$

$$u = x^2 + 2x \quad\rightarrow\quad dx\cdot\frac{du}{dx} = \left(2x+2\right)dx$$

$$du = \left(2x+2\right)dx$$

<u>Notes</u>

Guided Example

Use substitution to find the indefinite integral:

$$\int 5(5x+1)^4\,dx$$

Solution

$$\int 5(5x+1)^4\,dx = \int \underbrace{(5x+1)}_{u}{}^4\,\underbrace{5\,dx}_{du}$$

$$= \int u^4\,du$$

$$= \tfrac{1}{5}u^5 + C$$

$$= \tfrac{1}{5}(5x+1)^5 + C$$

$$u = 5x+1 \quad \rightarrow \quad dx \cdot \frac{du}{dx} = 5\,dx$$

$$du = 5\,dx$$

Practice

1. Use substitution to find the indefinite integral:

$$\int 2x\left(x^2+1\right)^3 dx$$

Guided Example

Use substitution to find the antiderivative

$$\int 7\left(x^2+2x\right)^6\left(x+1\right)dx$$

Solution Let $u=x^2+2x$ and $\dfrac{du}{dx}=2x+2$. If we multiply both sides by ½ on the derivative we get $\frac{1}{2}\dfrac{du}{dx}=x+1$. This means $\frac{1}{2}du=\left(x+1\right)dx$ and allows us to rewrite the right-hand side of the equation above as

$$\int 7\underbrace{\left(x^2+2x\right)}_{u}{}^6\underbrace{\left(x+1\right)dx}_{\frac{1}{2}du}=\frac{1}{2}\int 7u^6\,du$$

The antiderivative is found with the power rule as $\frac{1}{2}u^7+C$ so the final solution is

$$\int 7\left(x^2+2x\right)^6\left(x+1\right)dx=\frac{1}{2}\left(x^2+2x\right)^7+C$$

The $\frac{1}{2}$ in the antiderivative balances out the doubling we needed to do to introduce the correct du. Now more compactly:

$$\int 7\left(x^2+2x\right)^6\left(x+1\right)=\int 7\underbrace{\left(x^2+2x\right)}_{u}{}^6\underbrace{\left(x+1\right)dx}_{\frac{1}{2}du}$$

$$=\frac{1}{2}\cdot 7\int u^6du$$

$$=\frac{1}{2}\cdot 7\cdot\frac{1}{7}u^7+C$$

$$=\frac{1}{2}\left(x^2+2x\right)+C$$

$$u=x^2+2x \quad\rightarrow\quad \frac{1}{2}\cdot\frac{du}{dx}=\frac{1}{2}\left(2x+2\right)$$

$$dx\cdot\frac{1}{2}\cdot\frac{du}{dx}=\left(x+1\right)dx$$

$$\frac{1}{2}\cdot du=\left(x+1\right)dx$$

Practice

2. Use substitution to find the antiderivative

$$\int \left(x^3 + x^2\right)^3 \left(6x^2 + 4x\right) dx$$

Guided Example

Use substitution to find the indefinite integral:

$$\int \frac{6x+3}{\left(x^2+x\right)^5}\,dx$$

Solution

$$\int \frac{6x+3}{\left(x^2+x\right)^5}\,dx = \int \frac{1}{\underbrace{\left(x^2+x\right)}_{u}{}^5}\underbrace{(6x+3)\,dx}_{3du}$$

$$= 3\int \frac{1}{u^5}\,du$$

$$= 3\int u^{-5}\,du$$

$$= 3\frac{u^{-4}}{-4} + C$$

$$= -\frac{3}{4}\frac{1}{\left(x^2+x\right)^4} + C$$

$$u = x^2 + x \quad \rightarrow \quad 3\cdot\frac{du}{dx} = 3(2x+1)$$

$$dx\cdot 3\cdot\frac{du}{dx} = (6x+3)\,dx$$

$$3\cdot du = (6x+3)\,dx$$

Practice

3. Use substitution to find the indefinite integral:

$$\int \frac{10}{(5x-1)^2}\,dx$$

Guided Example

Evaluate the indefinite integral:

$$\int 4t\, e^{t^2}\, dt$$

Solution

$$\int 4t\, e^{t^2}\, dt = \int e^{\overbrace{t^2}^{u}} \cdot \underbrace{4t\, dt}_{2\,du}$$

$$= 2\int e^u\, du$$

$$= 2e^u + C$$

$$= 2e^{t^2} + C$$

$$u = t^2 \quad \rightarrow \quad 2 \cdot \frac{du}{dt} = 2 \cdot 2t$$

$$dt \cdot 2 \cdot \frac{du}{dt} = 4t\, dt$$

$$2 \cdot du = 4t\, dt$$

Practice

4. Evaluate the indefinite integral:

$$\int e^{2t+3}\, dt$$

Question 2 – How is the exact area under a function involving compositions computed?

Key Terms

Fundamental Theorem of Calculus

Summary

Area between a function and the x axis is calculated using a definite integral. We can use the Fundamental Theorem of Calculus to find the area,

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is the antiderivative of $f(x)$. If $f(x)$ involves compositions, substitution may be used to find the antiderivative. For instance, in the previous question we used u substitution to find the antiderivative

$$\int 7\left(x^2+2x\right)^6\left(2x+2\right)dx = \left(x^2+2x\right)^7 + C$$

If we want to evaluate the definite integral

$$\int_0^2 7\left(x^2+2x\right)^6\left(2x+2\right)dx$$

Since we know the antiderivative is $F(x) = \left(x^2+2x\right)^7 + C$, the value of the definite integral is

$$\begin{aligned}\int_0^2 7\left(x^2+2x\right)^6\left(2x+2\right)dx &= \left[\left(x^2+2x\right)^7 + C\right]_0^2 \\ &= \left(\left(2^2+2\cdot 2\right)^7 + C\right) - \left(\left(0^2+2\cdot 0\right)^7 + C\right) \\ &= 2097152 + C - C \\ &= 2097152\end{aligned}$$

Notes

Guided Example

Evaluate the definite integral:

$$\int_0^3 (x-1)\left(x^2-2x\right)^2\,dx$$

Solution Start by finding the antiderivative by u substitution:

$$\int (x-1)\left(x^2-2x\right)^2\,dx = \int \underbrace{\left(x^2-2x\right)}_{u}{}^2\,\underbrace{(x-1)\,dx}_{\frac{1}{2}du}$$

$$= \frac{1}{2}\int u^2 du$$

$$= \frac{1}{2}\frac{u^3}{3}+C$$

$$= \frac{1}{6}\left(x^2-2x\right)^3+C$$

$$u = x^2-2x \quad \rightarrow \quad \frac{1}{2}\cdot\frac{du}{dx} = \frac{1}{2}\cdot(2x-2)$$

$$dx\cdot\frac{1}{2}\cdot\frac{du}{dx} = (x-1)\,dx$$

$$\frac{1}{2}\cdot du = (x-1)\,dx$$

Now that we know the antiderivative, use the Fundamental Theorem of Calculus to find the definite integral,

$$\int_0^3 (x-1)\left(x^2-2x\right)^2\,dx = \left[\frac{1}{6}\left(x^2-2x\right)^3+C\right]_0^3$$

$$= \left(\frac{1}{6}\left(3^2-2\cdot 3\right)^3+C\right)-\left(\frac{1}{6}\left(0^2-2\cdot 0\right)^3+C\right)$$

$$= \frac{27}{6}+C-0-C$$

$$= \frac{27}{6}$$

Practice

1. Evaluate the definite integral:

$$\int_0^4 3(6x-1)^3 \, dx$$

Guided Example

Evaluate the definite integral:

$$\int_1^8 \frac{6(\ln x)^2}{x}\,dx$$

Solution Start with u substitution to find the antiderivative:

$$\int \frac{6(\ln x)^2}{x}\,dx = \int \underbrace{(\ln x)}_{u}{}^2 \cdot \underbrace{\frac{6}{x}dx}_{6du}$$

$$= 6\int u^2\,du$$

$$= 6\cdot\frac{u^3}{3} + C$$

$$= 2(\ln x)^3 + C$$

$$u = \ln x \quad \rightarrow \quad 6\cdot\frac{du}{dx} = 6\cdot\frac{1}{x}$$

$$dx\cdot 6\cdot\frac{du}{dx} = \frac{6}{x}dx$$

$$6\cdot du = \frac{6}{x}dx$$

Now use the Fundamental Theorem of Calculus to get the definite integral.

$$\int_e^8 \frac{6(\ln x)^2}{x}\,dx = \left[2(\ln x)^3 + C\right]_e^8$$

$$= \left(2(\ln 8)^3 + C\right) - \left(2(\ln e)^3 + C\right)$$

$$= 2(\ln 8)^3 + C - 2 - C$$

$$= 2(\ln 8)^3 - 2$$

Practice

2. Evaluate the definite integral:

$$\int_{2}^{4} \frac{2}{\left(x-1\right)^{2}}\,dx$$

Question 3 – How do you undo a rate with the Substitution Method?

Key Terms

Substitution method Fundamental Theorem of Calculus

Summary

If the integrand is given as a composition, we may need to use the Substitution Method to find the antiderivative of the rate. Undoing the rate will reveal the quantity associated with the rate. For instance, undoing the rate at which revenue changes (the marginal revenue) will lead to the revenue function. Applying the Fundamental Theorem of Calculus in this context allows us to compute the total change in revenue,

$$\int_a^b R'(x)\,dx = R(b) - R(a)$$

where $x = a$ and $x = b$ are two different production levels.

Notes

Guided Example

A company has found that the rate at which profit is changing (in millions of dollar per year) is given by the function

$$P'(t) = (8t+8)\left(t^2+2t+1\right)^{1/3}$$

a. Find the total profit over the first three years.

Solution To find the total profit over the first three years, we need to compute $P(3) - P(0)$. We can find $P(t)$ from the antiderivative of $P'(t)$. In fact, applying the Fundamental Theorem of Calculus, the total change in profit is computed from the definite integral,

$$\int_0^3 P'(t)\,dt = P(3) - P(0)$$

This is equivalent to

$$\int_0^3 (8t+8)\left(t^2+2t+1\right)^{1/3} dt = P(3) - P(0)$$

We will accomplish this by applying u substitution to the definite integral:

$$\begin{aligned}
\int (8t+8)\left(t^2+2t+1\right)^{1/3} dt &= \int \underbrace{\left(t^2+2t+1\right)}_{u}{}^{1/3} \underbrace{(8t+8)\,dt}_{4\,du} \\
&= 4\int u^{1/3}\,du \\
&= 4\cdot\frac{u^{4/3}}{4/3} + C \\
&= 3\left(t^2+2t+1\right)^{4/3} + C
\end{aligned}$$

Now apply the Fundamental Theorem of Calculus to give

$$\begin{aligned}
\int_0^3 (8t+8)\left(t^2+2t+1\right)^{1/3} dt &= \left[3\left(t^2+2t+1\right)^{4/3} + C\right]_0^3 \\
&= \left(3\left(3^2+2\cdot 3+1\right)^{4/3} + C\right) - \left(3\left(0^2+2\cdot 0+1\right)^{4/3} + C\right) \\
&= 3(16)^{4/3} + C - 3(1)^{4/3} - C \\
&\approx 117.95
\end{aligned}$$

in millions of dollars.

b. Find the profit in the fourth year of operation.

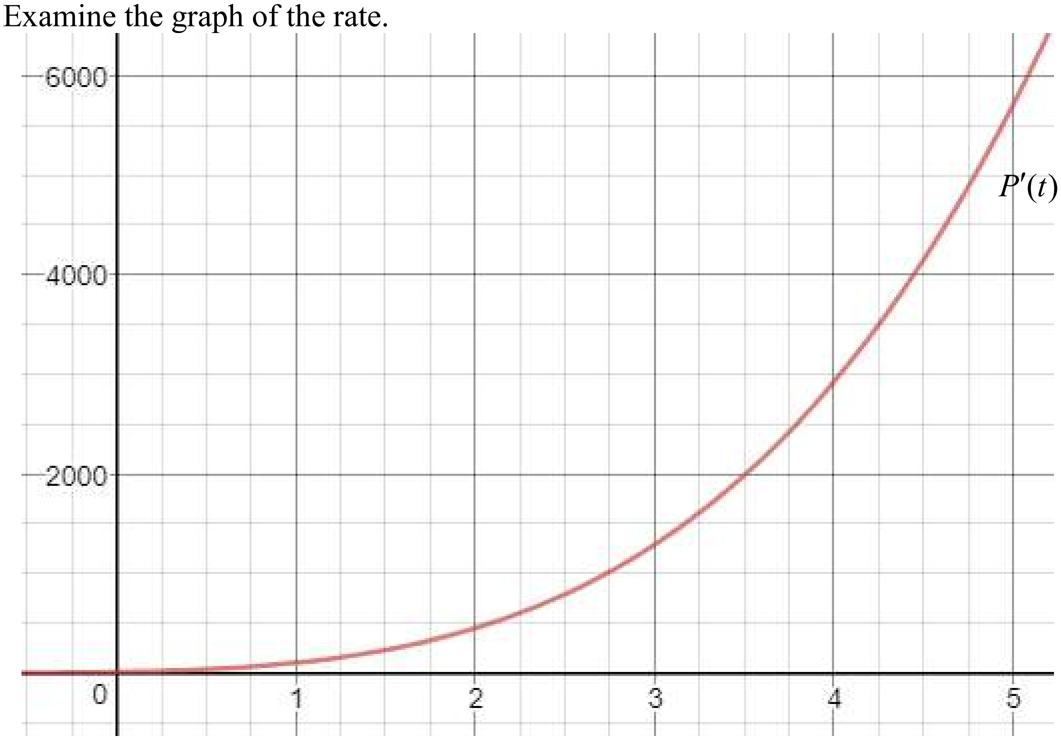
Solution

$$\int_3^4 (8t+8)\left(t^2+2t+1\right)^{1/3}\,dt = \left[3\left(t^2+2t+1\right)^{4/3}+C\right]_3^4$$
$$=\left(3\left(4^2+2\cdot 4+1\right)^{4/3}+C\right)-\left(3\left(3^2+2\cdot 0+1\right)^{4/3}+C\right)$$
$$=3(25)^{4/3}+C-3(16)^{4/3}-C$$
$$\approx 98.35$$

in millions of dollars.

c. What is happening to the annual profit over the long run?

Solution Examine the graph of the rate.



The annual areas from 1 to 2, 2 to 3, 3 to 4, … are getting larger and larger so the annual profit is getting larger and larger.

Practice

1. Suppose a company spends R million dollars on research annually. The rate $S'(R)$ at which sale are increasing at a company is given by

$$S'(R) = 50e^{0.05R} \text{ dollars of sales per dollars of research}$$

If the annual research budget is increased from 3 million dollars to 4 million dollars, how much will sales increase?

Section 14.2 Integration by Parts

Question 1 – How do we find the antiderivative of functions involving products?

Question 2 – How is the exact area under a function involving products computed?

Question 1 – How do we find the antiderivative of functions involving products?

Key Terms

Product

Summary

The Integration by Parts Rule allows us to take the antiderivative of functions that can be written as products.

Integration by Parts Rule

If u and v are differentiable functions,

$$\int uv'\,dx = uv - \int vu'\,dx$$

This rule requires us to match the integrand with the two pieces of the product, u and v'. Applying the rule results in a new antiderivative which should be easier to evaluate than the original antiderivative.

For example, consider the antiderivative $\int xe^x dx$ .To apply the Integration by Parts Rule, match u with x and v' with e^x. To find u', we need to take a derivative. To find v, we need to take an antiderivative.

$$u = x \xrightarrow{\text{Derivative}} u' = 1$$
$$v' = e^x \xrightarrow{\text{Antiderivative}} v = e^x$$

Put these pieces into the rule to give

$$\int \underbrace{x}_{u}\underbrace{e^x}_{v'}\,dx = \underbrace{x}_{u}\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\cdot\underbrace{1}_{u'}\,dx$$

The new integral is simply the antiderivative of e^x which is $e^x + C$. The original antiderivative may be evaluated as

$$\int xe^x\,dx = xe^x - e^x + C$$

We can abbreviate these steps as shown below:

$$\int \underbrace{x}_{u}\underbrace{e^x}_{v'}\,dx = \underbrace{x}_{u}\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\cdot\underbrace{1}_{u'}\,dx$$
$$= xe^x - e^x + C$$

$$u = x \xrightarrow{\text{Derivative}} u' = 1$$
$$v' = e^x \xrightarrow{\text{Antiderivative}} v = e^x$$

<u>Notes</u>

Guided Example

Use integration by parts to evaluate the integral

$$\int 8x\, e^{2x} dx$$

Solution

$$\int \underbrace{8x}_{u}\,\underbrace{e^{2x}}_{v'}\,dx = \underbrace{8x}_{u}\cdot\underbrace{\tfrac{1}{2}e^{2x}}_{v} - \int \underbrace{\tfrac{1}{2}e^{2x}}_{v}\cdot\underbrace{8}_{u'}\,dx$$

$$= 4xe^{2x} - 2e^{2x} + C$$

$$u = 8x \xrightarrow{\text{Derivative}} u' = 8$$

$$v' = e^{2x} \xrightarrow{\text{Antiderivative}} v = \tfrac{1}{2}e^{2x}$$

The new integral, $\int \frac{1}{2}e^{2x}\cdot 8\,dx$, is evaluated using the Substitution Rule:

$$\int \tfrac{1}{2}e^{2x}\cdot 8\,dx = 4\int e^{2x}\,dx$$

$$= 4\cdot\tfrac{1}{2}\int e^{u}\,du$$

$$= 2e^{u} + C$$

$$= 2e^{x} + C$$

$$u = 2x \quad\rightarrow\quad \frac{1}{2}\cdot\frac{du}{dx} = \frac{1}{2}\cdot 2$$

$$dx\cdot\frac{1}{2}\cdot\frac{du}{dx} = dx$$

$$\frac{1}{2}\cdot du = dx$$

Practice

1. Use integration by parts to evaluate the integral

$$\int 10x\, e^{4x} dx$$

Guided Example

Use integration by parts to evaluate the integral

$$\int 2x\ln(4x)\,dx$$

Solution

$$\int \underbrace{2x}_{v'}\,\underbrace{\ln(4x)}_{u}\,dx = \underbrace{\ln(4x)}_{u}\cdot\underbrace{x^2}_{v} - \int \underbrace{x^2}_{v}\cdot\underbrace{\tfrac{1}{x}}_{u'}\,dx$$

$$= \ln(4x)\cdot x^2 - \int x\,dx$$

$$= \ln(4x)\cdot x^2 - \frac{x^2}{2} + C$$

$$u = \ln(4x) \xrightarrow{\text{Derivative}} u' = \frac{1}{4x}\cdot 4 = \frac{1}{x}$$

$$v' = 2x \xrightarrow{\text{Antiderivative}} v = x^2$$

Practice

2. Use integration by parts to evaluate the integral

$$\int 3x^2\ln(10x)\,dx$$

Guided Example

Evaluate the integral

$$\int x\ln(\sqrt{x})\,dx$$

Solution The key to this problem is to rewrite the square root as a power and then to utilize the logarithm rule that allows powers to be brought out as a multiplier:

$$\int x\ln(\sqrt{x})\,dx = \int x\ln(x^{1/2})\,dx$$

$$= \int \tfrac{1}{2}x\ln(x)\,dx$$

Now we can apply integration by parts:

$$\int \underbrace{\ln(x)}_{u}\underbrace{\tfrac{1}{2}x}_{v'}\,dx = \underbrace{\ln(x)}_{u}\underbrace{\tfrac{x^2}{4}}_{v} - \int \underbrace{\tfrac{x^2}{4}}_{v}\cdot\underbrace{\tfrac{1}{x}}_{u'}\,dx$$

$$= \ln(x)\cdot\frac{x^2}{4} - \int \frac{x}{4}\,dx$$

$$= \ln(x)\cdot\frac{x^2}{4} - \frac{x^2}{8} + C$$

$$u = \ln(x) \xrightarrow{\text{Derivative}} u' = \frac{1}{x}$$

$$v' = \frac{1}{2}x \xrightarrow{\text{Antiderivative}} v = \frac{x^2}{4}$$

Practice

3. Evaluate the integral

$$\int 2x\ln(x^5)\,dx$$

Question 2 – How is the exact area under a function involving products computed?

Key Terms

Product Integration by Parts Fundamental Theorem of Calculus

Summary

When we find the exact area under a function by evaluating the definite integral, we utilize the Fundamental Theorem of Calculus. The Fundamental Theorem of Calculus helps us to evaluate the definite integral,

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is the antiderivative of $f(x)$. If $f(x)$ involves products, integration by parts may be used to find the antiderivative.

For instance, to find the area between $y = xe^x$ and the x axis from $x = 0$ to $x = 3$ we must evaluate the definite integral

$$\int_0^3 xe^x \, dx$$

In the previous question, we used integration by parts to find the antiderivative:

$$\int xe^x \, dx = xe^x - e^x + C$$

To evaluate the definite integral, we simply substitute x = 3 and x = 0 into the antiderivative and subtract,

$$\begin{aligned}\int_0^3 xe^x \, dx &= \left[xe^x - e^x + C \right]_0^3 \\ &= \left(3e^3 - e^3 + C\right) - \left(0e^0 - e^0 + C\right) \\ &= 2e^3 + 1 \\ &\approx 41.17\end{aligned}$$

Notes

Guided Example

Evaluate the definite integral

$$\int_{1}^{10} \ln(2x)\, dx$$

Solution To find the antiderivative of ln(2*x*), think of the integrand as the product of 1 and ln(2*x*). Then apply integration by parts to this product:

$$\int \underbrace{\ln(2x)}_{u} \cdot \underbrace{1}_{v'}\, dx = \underbrace{\ln(2x)}_{u}\, \underbrace{x}_{v} - \int \underbrace{x}_{v} \cdot \underbrace{\tfrac{1}{x}}_{u'}\, dx$$

$$= \ln(2x) \cdot x - \int 1\, dx$$

$$= \ln(2x) \cdot x - x + C$$

$$u = \ln(2x) \xrightarrow{\text{Derivative}} u' = \frac{1}{2x} \cdot 2 = \frac{1}{x}$$

$$v' = 1 \xrightarrow{\text{Antiderivative}} v = x$$

Note that the derivative of u is completed with the Chain Rule. Now that we have the antiderivative, we can apply the Fundamental Theorem of Calculus to evaluate the definite integral.

$$\int_{1}^{10} \ln(2x)\, dx = \left[\ln(2x) \cdot x - x + C\right]_{1}^{10}$$

$$= \left(\ln(20) \cdot 10 - 10 + C\right) - \left(\ln(2) \cdot 1 - 1 + C\right)$$

$$= \ln(20) \cdot 10 - 9 - \ln(2)$$

$$\approx 20.26$$

Practice

1. Evaluate the definite integral

$$\int_{1}^{3} \ln(5x)\, dx$$

Section 14.3 Area Between Curves

Question 1 – How is the area between two functions calculated?

Question 2 – What are consumers' and producers' surplus?

Question 1 – How is the area between two functions calculated?

<u>Key Terms</u>

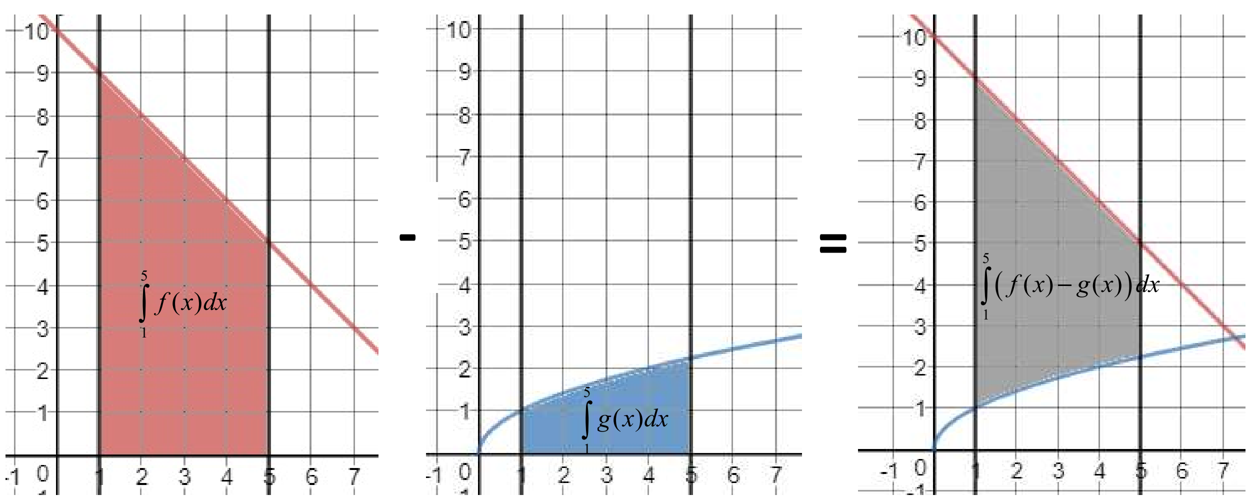
Region bounded by   Area between

<u>Summary</u>

To find the area between a function $f(x)$ and a function $g(x)$ over an interval from $x = a$ to $x = b$, we subtract the lower function from the higher function and calculate the definite integral from a to b,

$$\int_a^b \big(f(x) - g(x)\big)\,dx$$

To understand why this works, think about two functions $f(x)$ and $g(x)$ over an interval from $x = 1$ to $x = 5$. We can calculate the area under $f(x)$ and above the x axis with the definite integral $\int_1^5 f(x)dx$. The definite integral $\int_1^5 g(x)dx$ corresponds to the area under $g(x)$ and above the x axis. As shown in the graphic below, subtracting these areas yields the area between $f(x)$ and $g(x)$ from $x = 1$ to $x = 5$. The same area may be calculated by subtracting the functions first and then taking the definite integral from $x = 1$ to $x = 5$, $\int_1^5 \big(f(x) - g(x)\big)\,dx$.

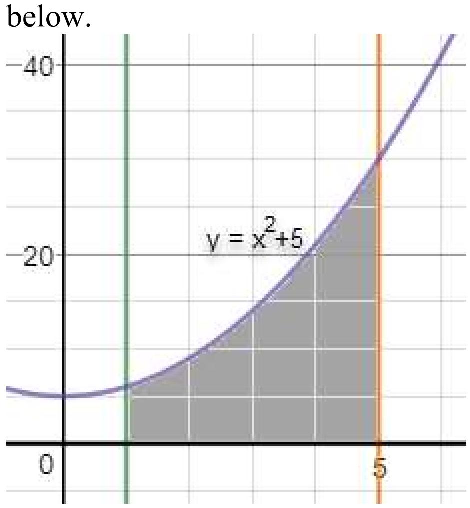


Notes

Guided Example

Find the area between $x = 1$, $x = 5$, $y = x^2 + 5$, and $y = 0$.

Solution The area enclosed by these curves are shown below.



This area corresponds to the definite integral

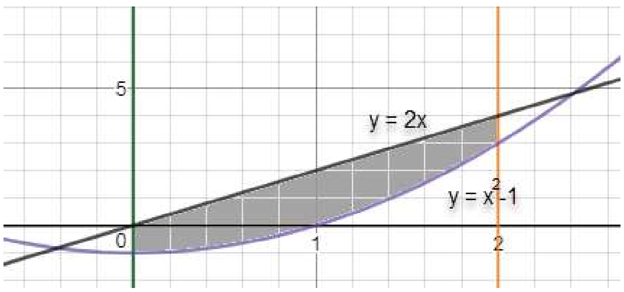
$$\int_1^5 \left(x^2 + 5\right) dx$$

Using the antiderivative and the Fundamental Theorem of Calculus, we get

$$\begin{aligned}\int_1^5 \left(x^2 + 5\right) dx &= \left[\frac{x^3}{3} + 5x\right]_1^5 \\ &= \left(\frac{5^3}{3} + 5(5)\right) - \left(\frac{1^3}{3} + 5(1)\right) \\ &= \frac{184}{3}\end{aligned}$$

Practice

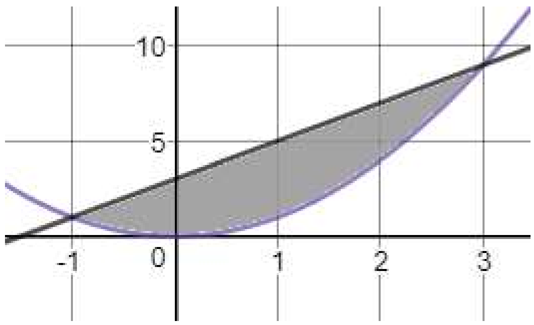
1. Find the area between $x = 2$, $x = 4$, $y = x^2 + 1$, and $y = 0$.

Guided Example	Practice
Find the area between $x=0$, $x=2$, $y=x^2-1$, and $y=2x$. **Solution** The area between these curves corresponds to the shaded region below.  The definite integral for this area is $$\int_0^2 \left(2x-\left(x^2-1\right)\right)dx$$ We can simplify the integrand and then use the Fundamental Theorem of Calculus to get $$\begin{aligned}\int_0^2 \left(2x-x^2+1\right)dx &= \left[x^2-\frac{x^3}{3}+x\right]_0^2 \\ &= \left(2^2-\frac{2^3}{3}+2\right)-\left(0^2-\frac{0^3}{3}+0\right) \\ &= \frac{10}{3}\end{aligned}$$	2. Find the area between $x=0$, $x=4$, $y=x^2+3$, and $y=-2x-1$.

Guided Example

Find the area of the region bounded by $y = x^2$ and $y = 2x + 3$.

Solution If we graph each curve, we see they enclose the area shown below.



It appears the curve cross at x = -1 and x = 3. We can conform this by setting the equations equal to each other and solve for x:

$$x^2 = 2x + 3$$
$$x^2 - 2x - 3 = 0$$
$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-3)}}{2(1)}$$
$$x = -1, 3$$

The definite integral for the shaded area is

$$\int_{-1}^{3} \left(2x + 3 - x^2\right) dx = \left[x^2 + 3x - \frac{x^3}{3} \right]_{-1}^{3}$$
$$= 9 - \left(-\frac{5}{3} \right)$$
$$= \frac{32}{3}$$

Practice

3. Find the area of the region bounded by $y = x^2 - 17x + 21$ and $y = -x^2 - 5x + 5$.

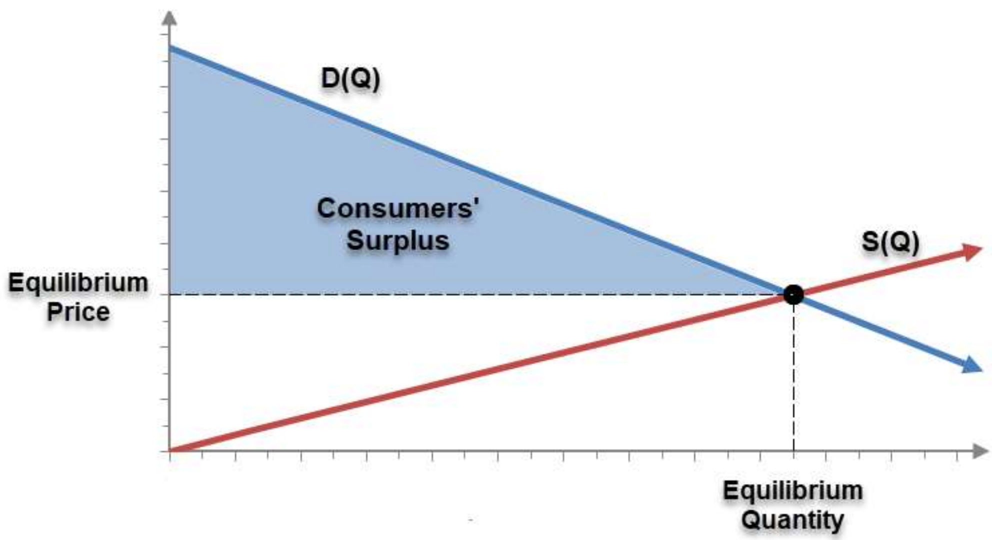
Question 2 – What are consumers’ and producers’ surplus?

Key Terms

Consumers’ surplus Producers’ surplus

Summary

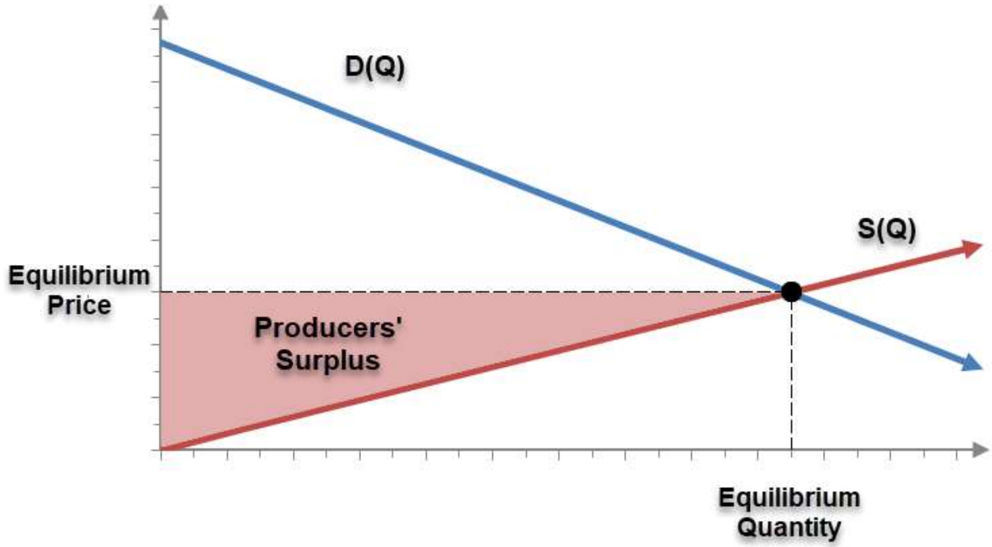
Consumers’ surplus indicates the additional money a consumer would save if prices follow the equilibrium price per unit instead of the price per unit on the demand function. On a graph of the demand function, this corresponds to the area under the demand function and above the equilibrium price.



The shaded area on the graph may be computed from the demand function $D(Q)$, the equilibrium price P_e, and the equilibrium quantity Q_e,

$$\text{Consumers' Surplus} = \int_0^{Q_e} \left(D(Q) - P_e \right) dQ$$

Producers’ surplus is the additional money a business would earn if prices follow the equilibrium price per unit instead of the price per unit on the supply function. On a graph of the supply function, this corresponds to the area under the equilibrium price and above the supply function.



The shaded area on the graph may be computed from the supply function $S(Q)$, the equilibrium price P_e, and the equilibrium quantity Q_e,

$$\text{Producers' Surplus} = \int_0^{Q_e} \left(P_e - S(Q)\right) dQ$$

<u>Notes</u>

Guided Example

Suppose the supply function for a product is

$$S(q) = 0.2q^2 \text{ dollars per unit}$$

and the demand function is

$$D(q) = -3q + 50 \text{ dollars per unit}$$

where q is the number of units of the product in thousands.

a. Find the equilibrium point.

Solution To find the equilibrium point, set the supply function equal to the demand function and solve for q.

$$0.2q^2 = -3q + 50$$

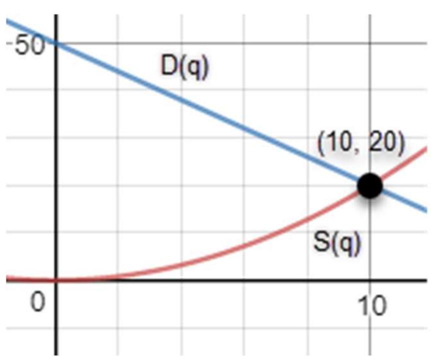
$$0.2q^2 + 3q - 50 = 0$$

$$q = \frac{-3 \pm \sqrt{3^2 - 4(0.2)(-50)}}{2(0.2)}$$

$$q = -25, 10$$

Only positive quantities make sense so q = 10 is the only reasonable equilibrium quantity. The corresponding equilibrium price is

$$D(10) = -3(10) + 50 = 20$$



Practice

1. Suppose the supply function for a product is

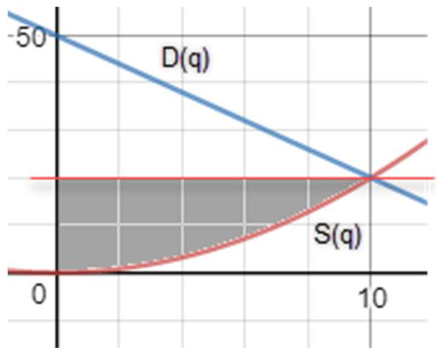
$$S(q) = 0.6q \text{ dollars per unit}$$

and the demand function is

$$D(q) = -0.28q^2 + 10 \text{ dollars per unit}$$

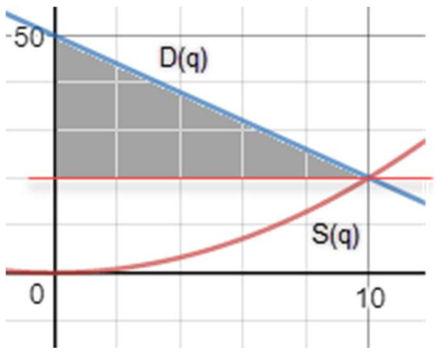
where q is the number of units of the product in thousands.

a. Find the equilibrium point.

b. Find the producers' surplus. **Solution** The producers' surplus is the area between $q = 0$, the equilibrium quantity, the equilibrium price, and the supply curve.  This is the area shaded in the graph above. To find this area, we need to evaluate the definite integral $$\int_0^{10} \left(20 - 0.2q^2\right) dq$$ Find the antiderivative and apply the Fundamental Theorem of Calculus to give $$\int_0^{10} \left(20 - 0.2q^2\right) dq = \left[20q - \tfrac{0.2}{3} q^3\right]_0^{10}$$ $$\approx 133.333$$ Area in this problem has dimensions of dollars per unit times thousands of units so it is in thousands of dollars. Thus the producers' surplus is approximately \$133,333.	b. Find the producers' surplus.

c. Find the consumers' surplus.	c. Find the consumers' surplus.

Solution The consumers' surplus is the area between $q = 0$, the equilibrium quantity, the demand curve, and the equilibrium price.



This is the area shaded in the graph above. To find this area, we need to evaluate the definite integral

$$\int_0^{10} \left(-3q + 50 - 20\right) dq$$

We can combine like terms in the integrand to yield

$$\int_0^{10} \left(-3q + 30\right) dq$$

Find the antiderivative and apply the Fundamental Theorem of Calculus to give

$$\int_0^{10} \left(-3q + 30\right) dq = \left[-\tfrac{3}{2}q^2 + 30q\right]_0^{10}$$

$$= 150$$

The consumers' surplus is \$150,000.

Chapter 14 Answers

Section 14.1

Question 1

1) $\dfrac{\left(x^2+1\right)^4}{4}+C$

2) $\dfrac{\left(x^3+x^2\right)^4}{2}+C$

3) $\dfrac{-2}{5x-1}+C$

4) $\tfrac{1}{2}e^{2t+3}+C$

Question 2

1) $\dfrac{1119363}{32}$

2) $\dfrac{4}{3}$

Question 3

1) $\int_3^4 50e^{0.05R}\,dR \approx 59.568515$ million dollars or \$59,568,515

Section 14.2

Question 1

1) $\dfrac{5}{2}xe^{4x}-\dfrac{5}{8}e^{4x}+C$

2) $x^3\ln(10x)-\dfrac{x^3}{3}+C$

3) $x^2\ln\left(x^5\right)-\dfrac{5}{2}x^2+C$

Question 2 1) $\ln(15) \cdot 3 - 2 - \ln(5) \approx 4.51$

Section 14.3

Question 1 1) $\frac{62}{3}$, 2) $\frac{160}{3}$, 3) $\frac{8}{3}$

Question 2 1) a. Equilibrium quantity is 5 thousand units and the equilibrium price is 3 dollars per unit.

b. Producers' surplus is 7.5 thousand dollars or \$7500.

c. Consumers' surplus is $\frac{70}{3}$ thousand dollars or approximately \$23,333.