Section 2.1 Systems of Linear Equations

Question 1 - What is a system of linear equations?

Question 2 - Where do systems of equations come from?

Question 1 - What is a system of linear equations?

Key Terms

Linear equation in *n* variables

System of linear equations in *n* variables

Solution

Point of intersection

Summary

A linear equation in n variables is any equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = k$$

where $a_1, a_2, \ldots, a_n$ and *k* are constants and $x_1, x_2, \ldots, x_n$ are variables. On the surface, this might look complicated, but it is very simple. On the left side, each term has a number multiplied by a variable to the first power. On the right side, there is a number. All of the below qualify as linear equations in some number of variables:

Two variables: $2x + 3y = 5, \quad p + 5q = 150$

Three variables: $x + 9y + 10z = 1, \quad 2x_1 + 6x_2 + x_3 = 10$

Four variables: $a - 5b + 2c - d = 12, \quad x_1 + x_2 + x_3 + x_4 = 100$

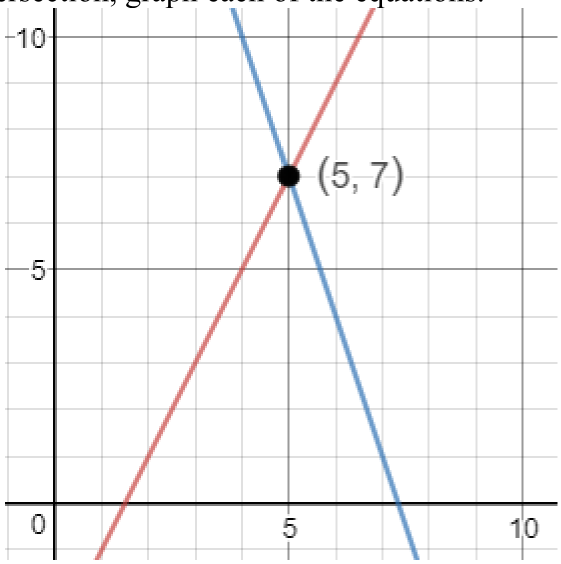
A system of linear equations in n variables is a finite number of equations where each equation can be written in the format above.

A solution to a system of linear equation is a collection of values, one for each variable, that makes all of the equation true. You can check to see if collection is a solution by substituting the values in place of the variables. If every equation in the system is true (has the same value on each side of the equation), then the collection is a solution. If we were to graph the equations, the solution will match the point at which all of the graphs intersect.

To find the point of intersection, you may need to solve the equations for the dependent variable so that you can use technology to graph the equation.

Notes

Guided Example	Practice
Determine if each ordered pair is a solution of the system of equations. $$\begin{aligned}2x+5y&=11\\4x-2y&=-14\end{aligned}$$ a. (3, 1) **Solution** To check to see whether an ordered pair is a solution, substitute the ordered pair into each equation. If you obtain the same value on each side of the equation in both equations, then the ordered pair is a solution to a system: $$\begin{aligned}2(3)+5(1)&=11\\4(3)-2(1)&=-14\end{aligned} \rightarrow \begin{aligned}11&=11\\10&\neq-14\end{aligned}$$ Since the second equation does not yield the same value on both sides, (3, 1) does not solve the system of equations. b. $(-2, 3)$ **Solution** Substitute the ordered pair into each equation: $$\begin{aligned}2(-2)+5(3)&=11\\4(-2)-2(3)&=-14\end{aligned} \rightarrow \begin{aligned}11&=11\\-14&=-14\end{aligned}$$ Since (-2, 3) yields the same value on both sides of the equation (in each equation), the ordered pair is a solution to the system of equations.	1. Determine if each ordered pair is a solution of the system of equations. $$\begin{aligned}x-3y&=-11\\4x+2y&=-2\end{aligned}$$ a. (2, -3) b. $(-2, 3)$

Guided Example	Practice
What are the coordinates of the point of intersection of $y = 2x - 3$ and $y = 22 - 3x$. **Solution** To find the coordinates of the point of intersection, graph each of the equations.  The point of intersection and solution to the system is (5, 7).	2. What are the coordinates of the point of intersection of $y = 3x + 2$ and $y = -3 - 2x$.

Guided Example

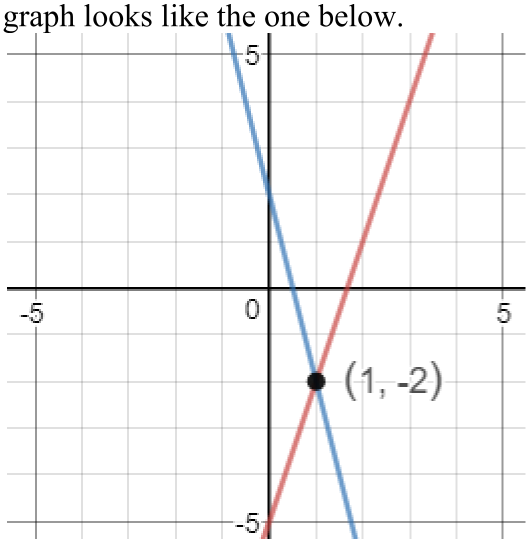
Solve the system of equations by graphing.

$$3x - y = 5$$
$$4x + y = 2$$

Solution As with the previous example, we need to graph each equation and locate the point of intersection. For many types of technology, we first need to solve each equation for y before we put in the equation:

$$y = 3x - 5$$
$$y = -4x + 2$$

The graph looks like the one below.



This leads to the solution $x = 1$ and $y = -2$.

Practice

3. Solve the system of equations by graphing.

$$6x + 2y = 10$$
$$3x - y = 7$$

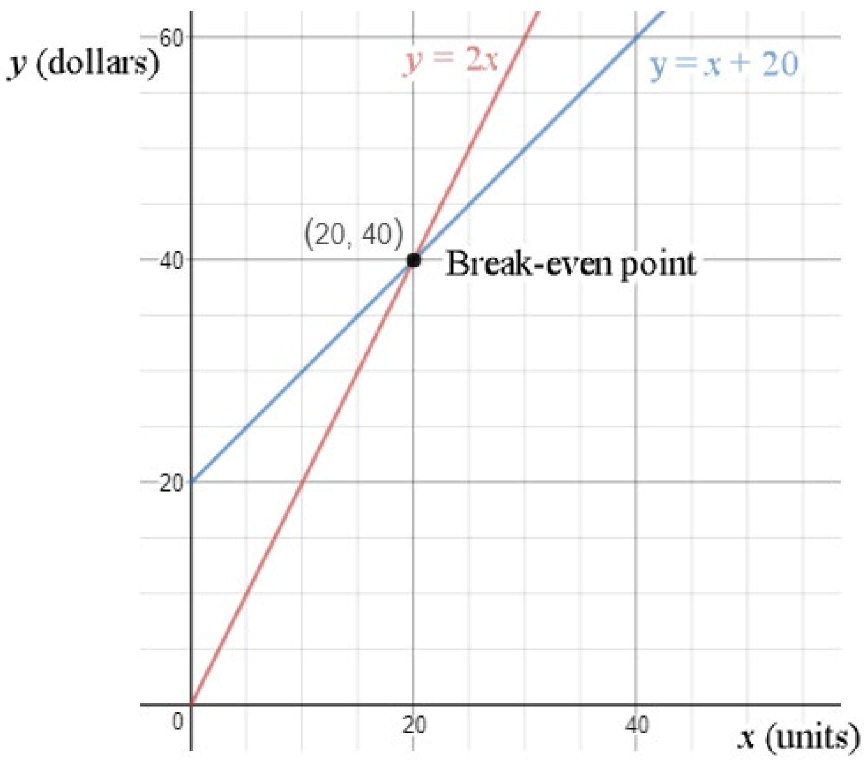
Question 2 - Where do systems of equations come from?

Key Terms

Revenue	Cost
Demand	Supply

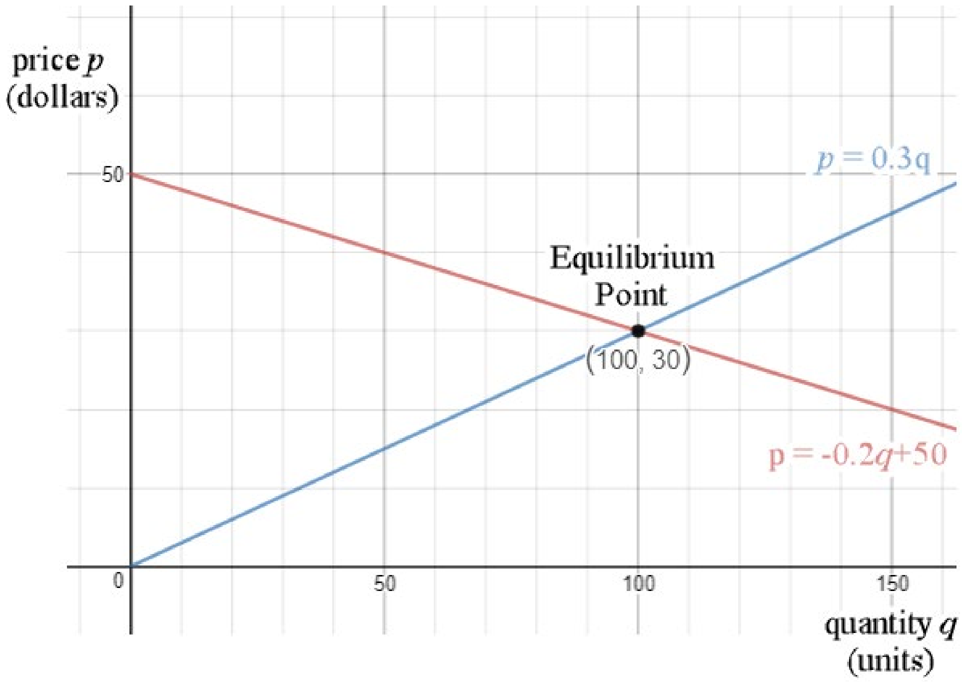
Summary

In business and economics, there are many applications that lead to systems of linear equations. For instance, suppose we are given an equation corresponding to revenue for a product and another equation corresponding to cost for the product. If both equations are linear equation, the equations form a system of linear equations.



The point where the graphs intersect is called the break-even point. At this quantity, the revenue from selling 20 units is equal to the cost of making 20 units.

Demand and supply equation result in a system of linear equations if each equation is a linear equation. On a graph of demand and supply, the point of intersection is called the equilibrium point. In the example below, a price of $30 corresponds to 30 units being demanded. Similarly, the price of $30 leads to 100 units being supplied. The equilibrium refers to the fact that the quantity demanded and supplied is the same.



price p
(dollars)
50
p = 0.3q
Equilibrium
Point
(100, 30)
p = -0.2q+50
0
50
100
150
quantity q
(units)

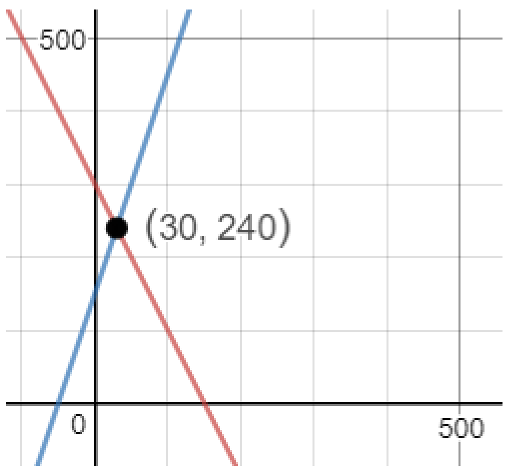
<u>Notes</u>

Guided Example

The demand for a stereo receiver is given by $p+2q=300$, and the supply for the stereo receiver is given by $p-3q=150$, where p is the price and q is the number of stereo in thousands.

a. Find the price at which the quantity demanded equals the quantity supplied.

Solution The point specified in the problem is the equilibrium point. To find this point, graph each equation (solve for p first if necessary) and then locate the point of intersection.



Since p is the dependent variable on this graph, the equilibrium price is \$240.

b. Find the equilibrium quantity.

Solution On the graph above, the quantity is the independent variable. At the point of intersection, the value of q is $q=30$ or 30,000 stereos.

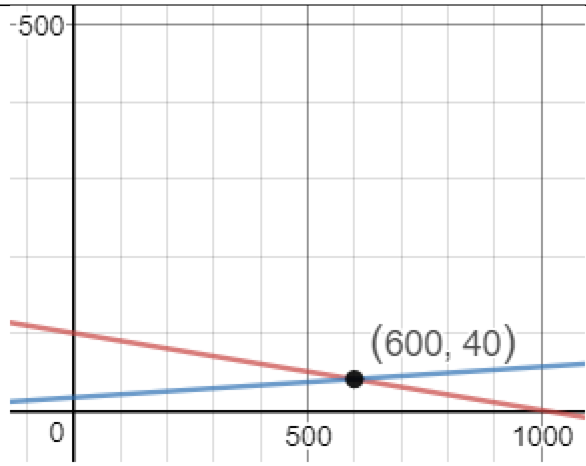
Practice

1. The demand for a DVD player is given by $p+3q=240$, and the supply for the DVD player is given by $p-2q=120$, where p is the price and q is the number of .DVD players in hundreds.

a. Find the price at which the quantity demanded equals the quantity supplied.

b. Find the equilibrium quantity.

Guided Example	Practice
A Verizon reseller will buy 700 phones if the price is \$30 each and 500 if the price is \$50. A wholesaler will supply 850 phones at \$50 each and 1100 at \$60 each. If the supply and demand functions are linear, find the market equilibrium point and explain what it means. **Solution** Before we find the point of intersection, we need to find the demand and supply functions. Let's start with the demand function. The information in the problem corresponding to demand are (700, 30) and (500, 50). The slope between the points is $$m=\frac{50-30}{500-700}=-0.1$$ Put this into a linear function of the form $p=mq+b$ to give $$p=-0.1q+b$$ Now put in one of the points and solve for b: $$\begin{aligned}50&=-0.1(500)+b\\50&=-50+b\\100&=b\end{aligned}$$ The demand equation is $p=-0.1q+100$. Follow the same procedure for the supply function but with the points (850, 50) and (1100, 60). This gives the supply function $$p=0.04q+16$$ Now graph the system of equations $$\begin{aligned}p&=-0.1q+100\\p&=0.04q+16\end{aligned}$$ and find the point of intersection.	2. A Verizon reseller will buy 1000 phones if the price is \$50 each and 900 if the price is \$55. A wholesaler will supply 600 phones at \$42.50 each and 1000 at \$66.50 each. If the supply and demand functions are linear, find the market equilibrium point and explain what it means.

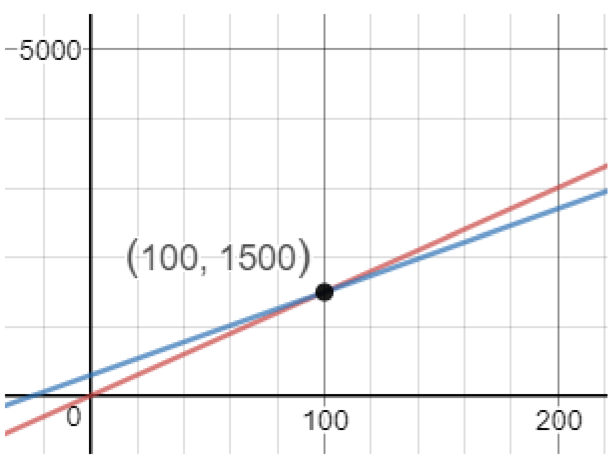


This point corresponds to an equilibrium quantity of 600 phones and an equilibrium price of $40.

Guided Example

A manufacturer has total revenue given by the function $R(x) = 15x$ and has total cost given by $C(x) = 12x + 300$ where x is the number of items produced and sold. Use graphical methods to find the number of units at the break-even point.

Solution The break-even point is the point of intersection of the two graphs. Graph both equations to give the graph below:



The break-even point corresponds to 100 units and $1500.

Practice

3. A manufacturer has total revenue given by the function $R(x) = 75x$ and has total cost given by $C(x) = 50x + 12500$ where x is the number of items produced and sold. Use graphical methods to find the number of units at the break-even point.

Section 2.2 Solving a System of Two Linear Equations Algebraically

Question 1 - How do you solve a system of linear equations?

Question 1 - How do you solve a system of linear equations?

Key Terms

Substitution Method Elimination Method

Equivalent system

Summary

The Substitution and Eliminatio0n Methods are techniques which changes the system of equations to an equivalent system of equations. Equivalent systems are sytems that have the same solutions.

The Substitution Method is a technique for finding the solution to a system of equations.

Substitution Method

1. Solve for one of the variables in one of the equations. If it is difficult to solve for a variable, the Elimination Method may be better suited to solve the system.
2. In the other equation, replace the variable you solved for in step 1 with the equivalent expression. Once you have replaced the variable in the other equation, there should only be one variable in this equation.
3. Solve the equation containing only one variable for that variable.
4. To find the value of the other variable, place the value obtained in step 3 into the equation from step 1.

This method is very useful when it is easy to solve for a variable in one of the equations.

The Elimination Method is a technique for solving systems of linear equations. In this technique, we multiply equations by a constant and then add them to eliminate a variable.

Elimination Method

1. Write each equation in the system with the variables on the left side and the constants on the right side of the equation.
2. Rearrange the terms on the left side of the equation so that the variables appear in the same order in each equation. Write the terms so that each term with a specific variable is vertically align with terms containing the same variable.
3. Multiply the first equation by the reciprocal of the coefficient of the first term (the leading coefficient). After this transformation, the coefficient of the first variable in the first equation should be a 1.
4. Eliminate the first variable from all equations except the first equation using equation transformations.

5. Multiply the second equation by the reciprocal of the leading coefficient. After this transformation the leading coefficient of second equation should be a 1.
6. Eliminate the variable corresponding to the leading coefficient from all other equations except for the second equation.
7. Continue this process for each equation and leading coefficient.
8. Solve each equation for the leading variable to yield the solution to the system of equations.

<u>Notes</u>

Guided Example	Practice
Solve the system of equations by the Substitution Method. $$x = 2y + 1$$ $$4x - 3y = 14$$ **Solution** Since the first equation is already solved for *x*, replace the *x* in the second equation with $2y + 1$ and solve for *y*: $$4(2y+1) - 3y = 14$$ $$8y + 4 - 3y = 14$$ $$5y + 4 = 14$$ $$5y = 10$$ $$y = 2$$ Now substitute this value into the first equation: $$x = 2(2) + 1 = 5$$ The solution to the system is $(x, y) = (5, 2)$.	1. Solve the system of equations by the Substitution Method. $$y = -2x + 3$$ $$2x - 5y = -51$$

Guided Example	Practice
Solve the system of equations by the Substitution Method. $$2x-3y=4$$ $$5x+2y=29$$ **Solution** To carry out the Substitution Method, we need to solve for a variable in one of the equations. It does not matter which variable we pick or which equation we use. For no particular reason, solve for x in the first equation: $$2x=3y+4$$ $$x=\tfrac{3}{2}y+2$$ Now put $\frac{3}{2}y+2$ in place of x in the second equation and solve for y: $$5\left(\tfrac{3}{2}y+2\right)+2y=29$$ $$\tfrac{15}{2}y+10+2y=29$$ $$\tfrac{19}{2}y+10=29$$ $$\tfrac{19}{2}y=19$$ $$y=2$$ Now find the x value from the original substitution: $$x=\tfrac{3}{2}(2)+2=5$$ The solution to the system of equations is $(x,y)=(5,2)$.	2. Solve the system of equations by the Substitution Method. $$4x-9y=-1$$ $$3x+4y=10$$

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<tr><th>Guided Example</th><th>Practice</th></tr>
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Solve the system of equations by the Elimination Method.

$$2x+3y=-1$$
$$7x-6y=13$$

Solution To carry out elimination method, multiply the first equation by $\frac{1}{2}$. This gives an equivalent system

$$x+\tfrac{3}{2}y=-\tfrac{1}{2}$$
$$7x-6y=13$$

To eliminate x in the second equation, multiply the first equation by -7 and add it to the second equation

$$-7x-\tfrac{21}{2}y=\tfrac{7}{2}$$
$$\underline{7x-6y=13}$$
$$-\tfrac{33}{2}y=\tfrac{33}{2}$$

which leads to $y=-1$.

To find the value for y, substitute the value for x into one of the original equations and solve for y:

$$7x-6(-1)=13$$
$$7x+6=13$$
$$7x=7$$
$$x=1$$

The solution to the system of equations is $(x,y)=(1,-1)$.
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3. Solve the system of equations by the Elimination Method.

$$-2x-5y=18$$
$$10x+2y=2$$
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Guided Example	Practice
Solve the system of equations by the Elimination Method. $$0.1x + 0.3y = 1.4$$ $$9x - 3y = -24$$ **Solution** For many students, the decimals in the first equation are intimidating. Let's get rid of those decimals by multiplying the first equation by 10. This gives an equivalent system of equations $$x + 3y = 14$$ $$9x - 3y = -24$$ We can eliminate y by adding the equations together (or multiply the first equation by -9 and add the equations): $$\begin{aligned} x + 3y &= \ \ 14 \\ \underline{9x - 3y} &\underline{= -24} \\ 10x \qquad &= -10 \end{aligned}$$ The solution for x is $x = -1$. Put this value into one of the original equations and solve for y: $$\begin{aligned} 9(-1) - 3y &= -24 \\ -9 - 3y &= -24 \\ -3y &= -15 \\ y &= 5 \end{aligned}$$ The solution to the system is $(x, y) = (-1, 5)$.	4. Solve the system of equations by the Elimination Method. $$0.5x - 0.25y = -0.5$$ $$-2x + 3y = 10$$

Guided Example	Practice
The demand for a certain product is given by $p+4q=334$, and the supply for this product is given by $p-6q=74$ where p is the price in dollars and q is the number of products in hundreds. Find the price and quantity at which the quantity demanded equals the quantity supplied. **Solution** Let's solve the system $p+4q=334$ $p-6q=74$ by the Elimination Method. Multiply the first equation by -1 and add it to the second equation: $-p-4q=-334$ $p-6q=\ \ 74$ $-10q=-260$ We can solve the resulting equation for q to give $q=26$. This corresponds to a quantity of 2600 products. To find the corresponding price, put the value for q into one of the original equations and solve for p: $p-6(26)=74$ $p-156=74$ $p=230$ The equilibrium price is \$230.	5. The demand for a certain product is given by $p+2q=200$, and the supply for this product is given by $p-2q=0$ where p is the price in dollars and q is the number of products in thousands. Find the price and quantity at which the quantity demanded equals the quantity supplied.

Section 2.3 Systems with Many or No Solutions

Question 1 - Does every system have a unique solution?

Question 2 - How do you solve a system of three equations in two variables?

Question 3 - How do you set up and solve an application involving systems of equations in two variables?

Question 1 - Does every system have a unique solution?

Key Terms

Inconsistent Dependent

Summary

The systems we encountered in Section 2.2 had a unique solution. This means that there was only one combination of variables that solved the system of equation. In this section, we will examine two other possibilities for systems of linear equations.

An inconsistent system of linear equations has no solutions. This means there is no combination of values for the variables that make all of the equations in the system true. This type of system is signaled by a contradiction in the process of solving the system of equations. A contradiction is simply an equation that cannot be true.

A dependent system of linear equations has many solutions. This means there are many combinations of the variables that make all of the equations in the system true. Dependent equations are signaled by an identity. This happens in the process of solving the system when we get an equation that is always true

Notes

Guided Example	Practice
Solve the system of two equations in two unknowns. $$x = 1 - 2y$$ $$3x + 6y = 10$$ **Solution** Since the first equation is solved for a variable, use the Substitution Method and replace x with $1 - 2y$ in the second equation: $$3(1-2y) + 6y = 10$$ $$3 - 6y + 6y = 10$$ $$3 = 10$$ Since this is a contradiction or false statement (3 cannot equal 10), the system is inconsistent and has no solutions.	1. Solve the system of two equations in two unknowns. $$y = 5x + 2$$ $$-10x + 2y = 1$$

Guided Example	Practice
Solve the system of two equations in two unknowns. $$5x + 6y = 1$$ $$15x + 18y = 3$$ **Solution** Since these equations are more difficult to solve for a variable, let's apply the Elimination Method to solve the system. Start by multiplying the first equation by $\frac{1}{5}$. This results in an equivalent system of equations, $$x + \tfrac{6}{5}y = \tfrac{1}{5}$$ $$15x + 18y = 3$$ In the next step, we need to eliminate the term with x in the second equation by multiply the first equation by -15 and add it to the second equation: $$\begin{array}{r} -15x - 18y = -3 \\ 15x + 18y = 3 \\ \hline 0 = 0 \end{array}$$	2. Solve the system of two equations in two unknowns. $$2x - 3y = 5$$ $$8x - 12y = 20$$

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This gives us an equivalent system of equations:

$$\begin{aligned} x + \tfrac{6}{5}y &= \tfrac{1}{5} \\ 0 &= 0 \end{aligned}$$

Since the second equation is always true, this system of equations has many solutions. To find these solutions, solve the first equation for *x*:

$$x = -\tfrac{6}{5}y + \tfrac{1}{5}$$

This equation gives us a recipe for finding solutions to the system. The solutions are ordered pairs of the form

$$\left(-\tfrac{6}{5}y + \tfrac{1}{5}, y\right)$$

where y can be any number.
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Guided Example

For the following system of equations, tell how many solutions there are that are positive and result in *z* being a positive integer.

$$\begin{aligned} x + 2y + z &= 88 \\ 7y + 3z &= 84 \end{aligned}$$

Solution Start by multiplying the second equation by $\tfrac{1}{7}$ to make the leading coefficient a 1. This gives the equivalent system

$$\begin{aligned} x + 2y + z &= 88 \\ y + \tfrac{3}{7}z &= 12 \end{aligned}$$

To eliminate the term with y in the first equation, multiply the second equation by -2 and add to the first equation:

$$\begin{array}{rrrl} x + & 2y + & z = & 88 \\ & -2y - & \tfrac{6}{7}z = & -24 \\ \hline x & + & \tfrac{1}{7}z = & 64 \end{array}$$

The equivalent system of equations is

$$\begin{aligned} x \quad + \tfrac{1}{7}z &= 64 \\ y + \tfrac{3}{7}z &= 12 \end{aligned}$$

Solve each equation for the leading variable to give

$$\begin{aligned} x &= -\tfrac{1}{7}z + 64 \\ y &= -\tfrac{3}{7}z + 12 \end{aligned}$$

To get an idea of where nonnegative solutions begin, set each equation equal to zero and solve for z:

$$\begin{array}{ll} 0 = -\tfrac{1}{7}z + 64 & 0 = -\tfrac{3}{7}z + 12 \\ \tfrac{1}{7}z = 64 & \tfrac{3}{7}z = 12 \\ z = 448 & z = 28 \end{array}$$

Let's use these values along with $z = 0$ and compute complete solutions.

z	0	21	28	35	448
$x = -\frac{1}{7}z + 64$	64	61	60	59	0
$y = -\frac{3}{7}z + 12$	12	3	0	-3	-180

As z increases, both x and y decrease until y becomes 0 at $z = 28$ So positive solutions correspond to $z = 1, \ldots, 27$. There are 27 positive solutions. If the question had requested nonnegative integers, we would also include the solutions at $z = 0$ and $z = 28$ since these lead to zero values for the variables.

Practice

3. For the following system of equations, tell how many solutions there are in nonnegative integers.

$$\begin{aligned} x + 3y + z &= 82 \\ 7y + 2z &= 42 \end{aligned}$$

Guided Example

Practice

Guided Example	Practice
Solve the following system of equations. Let z be the parameter. $$x+3y+6z=19$$ $$2x-3y+3z=2$$ **Solution** To eliminate x in the second equation multiply the first equation by -2 and add it to the second equation:	4. Solve the following system of equations. Let z be the parameter. $$x+2y+3z=1$$ $$2x+y-z=6$$

$$\begin{array}{r} -2x-6y-12z=-38 \\ 2x-3y+\ \ 3z=\ \ \ 2 \\ \hline -9y-\ \ 9z=-36 \end{array}$$

This gives an equivalent system

$$\begin{aligned} x+3y+6z&=19 \\ -9y-9z&=-36 \end{aligned}$$

To make the leading coefficient in the second equation a 1, multiply the second equation by $-\frac{1}{9}$

$$\begin{aligned} x+3y+6z&=19 \\ y+\ \ z&=4 \end{aligned}$$

Now eliminate the *y* term in the first equation by multiplying the second equation by -3 and add to the first equation:

$$\begin{array}{r} x+3y+6z=19 \\ -3y-3z=-12 \\ \hline x\qquad +3z=7 \end{array}$$

Our system of equations becomes

$$\begin{aligned} x\quad +3z&=7 \\ y+\ \ z&=4 \end{aligned}$$

Solving for the leading variables give the solution

$$\begin{aligned} x&=-3z+7 \\ y&=-z+4 \end{aligned}$$

Giving the ordered triple $(-3z+7,-z+4,z)$.

Question 2 - How do you solve a system of three equations in two variables?

Key Terms

Summary

To solve any system, we follow the same procedure as before in which we make leading coefficients equal to 1 and then eliminate the leading variable in the other equations.

Notes

Guided Example	Practice
Solve the following system of equations. $$\begin{aligned} x-3y&=-1\\ x+y&=7\\ 2x+5y&=20 \end{aligned}$$ **Solution** Follow the process of making leading coefficients 1 and then eliminate variables above and below the leading coefficient. For this system, we start by multiplying the first equation by -1 and add it to the second equation to yield a new second equation $$\begin{array}{r} -x+3y=1\\ x+\quad y=7\\ \hline 4y=8 \end{array}$$ To eliminate x from the third equation, multiply the first equation by -2 and add it to the third equation $$\begin{array}{r} -2x+6y=2\\ 2x+5y=20\\ \hline 11y=22 \end{array}$$ Putting these equation in place of the second and third equation gives $$\begin{aligned} x-3y&=-1\\ 4y&=8\\ 11y&=22 \end{aligned}$$ Multiply the second equation by $\frac{1}{4}$: $$\begin{aligned} x-3y&=-1\\ y&=2\\ 11y&=22 \end{aligned}$$ Eliminate y from the first equation by multiplying the second equation by 3 and add it to the first equation:	1. Solve the following system of equations. $$\begin{aligned} x-y&=2\\ x+2y&=11\\ -2x+5y&=5 \end{aligned}$$

$$\begin{array}{r} x-3y=-1 \\ 3y=6 \\ \hline x \quad\quad =5 \end{array}$$ Eliminate y from the third equation by multiplying the second equation by -11 and add it to the third equation: $$\begin{array}{r} -11y=-22 \\ 11y=22 \\ \hline 0=0 \end{array}$$ Now put in these new equations to give $$\begin{aligned} x \quad &= 5 \\ y &= 2 \\ 0 &= 0 \end{aligned}$$ The last equation initially indicates an infinite number of solutions. However, the other two equations show that only the solution $(5,2)$ solves all three equations.	

Question 3 - How do you set up and solve an application involving systems of equations in two variables?

Key Terms

Summary

Let's look at a simple example which we can set up and solve.

A restaurant owner orders a replacement set of knives, fork, and spoons. The box arrives containing 40 utensils and weighing 141.3 ounces (ignoring the weight of the box). A knife, fork, and spoon weigh 3.9 ounces, 3.6 ounces, and 3.0 ounces, respectively.

How many knives, forks and spoons are in the box?

Since we are being asked to find the number of knives, forks, and spoons, let's make the following designations:

K: number of knives

F: number of forks

S: number of spoons

The first thing to notice is that you are given a total number of utensils (40) and a total weight for the utensils (141.3 ounces). These are the prime candidates for writing out the equations.

Starting with

$$\text{Total number of utensils} = 40$$

It is fairly obvious that

$$K + F + S = 40$$

Starting with

$$\text{Total weight of utensils} = 141.3$$

We can deduce that the individual weights are

$$\text{Weight of knives} = 3.9\,K$$

$$\text{Weight of forks} = 3.6\,F$$

$$\text{Weight of spoons} = 3.0\,S$$

So

$$3.9\,K + 3.6\,F + 3.0\,S = 141.3$$

Now what? Your experience in these sections probably tells you that you need another equation in the three unknowns to be able to solve for *K*, *F*, and *S*. This is true if there is a unique solution to this problem. But this problem has many possible solutions (ie. There are many ways to have 40 utensils that weigh 141.3 ounces). Now let's solve the system.

Start by multiplying the first equation by -3.9 and add it to the second.

$$\begin{array}{r} -3.9K - 3.9F - 3.9S = -156 \\ 3.9K + 3.6F + 3.0S = 141.3 \\ \hline -0.3F - 0.9S = -14.7 \end{array}$$

Place the sum in place of the second equation to yield

$$\begin{aligned} K + \quad F + \quad S &= \quad 40 \\ -0.3F - 0.9S &= -14.7 \end{aligned}$$

Next we place a 1 in front of b in the second equation. This is done by multiplying the second equation by $-\frac{1}{0.3}$:

$$\begin{array}{r} -0.3F - 0.9S = -14.7 \\ \times -\frac{1}{0.3} \\ \hline F + \quad 3S = 49 \end{array}$$

Place the result in place of the second equation to give the equivalent system,

$$\begin{aligned} K + \quad F + \quad S &= 40 \\ F + \quad 3S &= 49 \end{aligned}$$

Multiply the second equation by -1 and add it to the first equation:

$$\begin{array}{r} -F - 3S = -49 \\ K + F + \ S = \ \ 40 \\ \hline K \qquad -2S = -9 \end{array}$$

Replace the first equation with this sum to yield

$$\begin{aligned} K + \quad - \quad 2S &= -9 \\ F + \quad 3S &= 49 \end{aligned}$$

Solve the first equation for a and the second equation for b. This gives

$$K = -9 + 2S$$
$$F = 49 - 3S$$

The original system has an infinite number of solutions. Each solution corresponds to a different value for *c*. Each solution has the form

$$(-9 + 2S, 49 - 3S, S)$$

The variable *S* can be any value that makes sense for the problem. For instance, we certainly know that *S* (the number of spoons) should be non-negative integers like $S = 0, 1, 2, \ldots$ If we make up a table, we can make some interesting observations:

Spoons	S	0	1	2	5	6	7	14	15	16	17
Knives	$2S - 9$	-9	-7	-5	1	3	5	19	21	23	25
Forks	$-3S + 49$	49	46	43	34	31	28	7	4	1	-2

We need the numbers of each type of utensil to be positive…to do that we'll require the number of spoons to be $S = 5, 6, 7, \ldots, 16$. So even though the system has an infinite number of solutions, only certain ones make sense.

Notes

Guided Example | Practice

Guided Example	Practice
A trust account manager has \$420,000 to be invested in three different accounts. The accounts pay 8%, 10%, and 12%, and the goal is to earn 44,000 with minimum risk. How much can be invested in each account with the largest possible amount invested at 8%? **Solution** Define the variables first: x: amount invested at 8% y: amount invested at 10% z: amount invested at 12% Since there is \$420,000 to be invested, $x+y+z=420000$ Each account earns 0.08x, 0.10y, and 0.12z so the total earnings tell us that $0.08x+0.10y+0.12z=44000$ We now have the system of equations $\begin{aligned} x+\quad y+\quad z&=420000 \\ 0.08x+0.10y+0.12z&=44000 \end{aligned}$ To solve this system, multiply the first equation by -0.08 and add it to the second equation: $\begin{array}{r} -0.08x-0.08y-0.08z=-33600 \\ 0.08x+0.10y+0.12z=44000 \\ \hline 0.02y+0.04z=10400 \end{array}$ Put this equation in place of the second equation: $\begin{aligned} x+\quad y+\quad z&=420000 \\ 0.02y+0.04z&=10400 \end{aligned}$ Multiply the second equation by $\frac{1}{0.02}$:	1. Katherine Chong invests \$10,000 received from her grandmother in three ways. With one part, she buys US savings bonds at an interest rate of 2.5% per year. She uses the second part to buy mutual funds that offer a return of 6% per year. She puts the rest of the money into a money market account paying 4.5% annual interest. The first year her investments bring a return of \$470. How much did she invest in each way if she put the largest amount in mutual funds?

$$\begin{aligned} x + y + \quad z &= 420000 \\ y + 2z &= 520000 \end{aligned}$$ Eliminate y in the first equation by multiplying the second equation by -1 and adding it to the first equation: $$\begin{array}{r} x + y + \quad z = 420000 \\ \underline{-y - 2z = -520000} \\ x \qquad - z = -100000 \end{array}$$ Our system is now $$\begin{aligned} x \quad - \quad z &= -100000 \\ y + 2z &= 520000 \end{aligned}$$ The solution is $$\begin{aligned} x &= z - 100000 \\ y &= -2z + 520000 \end{aligned}$$ The variable x is equal to zero when $z = 100000$. At this point, $y = 320000$. We are interested in having the amount in x as large as possible. This will occur when y is zero. Set the second equation equal to zero and solve for z: $$\begin{aligned} 0 &= -2z + 520000 \\ 2z &= 520000 \\ z &= 260000 \end{aligned}$$ At this amount, the amount in x is $$x = 260000 - 100000 = 160000$$ Giving us the solution $$(x, y, z) = (160000, 0, 260000)$$	

Section 2.4 Solving a System of Linear Equations with Matrices

Question 1 - What is a matrix?

Question 2 - How do you form an augmented matrix from a system of linear equations?

Question 3 - How do you use row operations to determine the reduced row echelon form of a matrix?

Question 4 - Do all systems of linear equations have unique solutions?

Question 5 - How do you mix different grades of ethanol to create a new grade of ethanol?

Question 1 - What is a matrix?

Key Terms

Matrix

Summary

A matrix is a table of numbers organized into rows and columns. Matrices are symbolized with capital letters. An example would be the matrix A below:

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 6 & 4 & 5 \end{bmatrix}$$

The entries in a matrix are placed within brackets. The size of a matrix is described by the number of rows and columns in the matrix. The matrix A above is a 2×3 matrix (read 2 by 3) since it has 2 rows and 3 columns. When quoting the size of a matrix, the number of rows are listed first and the number of columns is listed second.

We can refer to the entries using lower case letters and subscripts. The notation a_{mn} is the entry in the matrix A in the m^{th} row and n^{th} column. In the matrix above, $a_{21} = 6$ is the entry in the second row and third column.

Notes

Guided Example	Practice
Suppose you are given the matrix $$A = \begin{bmatrix} 1 & -2 \\ 3 & 0 \\ -1 & 5 \end{bmatrix}$$	1. Suppose you are given the matrix $$B = \begin{bmatrix} -1 & 3 & -2 \\ 2 & 1 & -3 \\ 0 & 4 & 5 \end{bmatrix}$$
a. Find the size of the matrix.	a. Find the size of the matrix.
Solution The matrix has three rows and two columns, so it is a 3 × 2 matrix.	
b. Find the value of a_{12} .	b. Find the value of b_{23} .
Solution The entry a_{12} is the entry in the first row and second column so $a_{12} = -2$.	
c. Find the value of a_{32} .	c. Find the value of b_{31} .
Solution The entry a_{32} is the entry in the third row and second column so $a_{12} = 5$.	

Question 2 - How do you form an augmented matrix from a system of linear equations?

Key Terms

Augmented matrix

Summary

An augmented matrix is a matrix whose entries correspond to a system of linear equations. For instance, if we have the system of linear equations,

$$\begin{aligned} x+2y+3z&=10 \\ -x+4y-2z&=3 \\ 2y+\ \ z&=5 \end{aligned}$$

the augmented matrix corresponding to the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ -1 & 4 & -2 & 3 \\ 0 & 2 & 1 & 5 \end{array}\right]$$

Comparing the system with the matrix. We see that the fourth column corresponds to the constants on the right side of the equal signs in the system. The vertical line corresponds to the equal signs in the system. Each column in the matrix matches the coefficients on the variables in the system. Also note that all terms with variable are on one side of the equation and the constants are on the opposite side.

Notes

Guided Example	Practice
Write an augmented matrix for each of the system of equations below.	1. Write an augmented matrix for each of the system of equations below.
a. $\begin{aligned} 2x+3y&=9 \\ x-\ \ y&=-1 \end{aligned}$	a. $\begin{aligned} -5x+2y&=12 \\ 3x-4y&=1 \end{aligned}$
Solution The variable terms are on one side of the equal sign and the constants are on the other side. Using the coefficients and the constants gives us the augmented matrix $$\left[\begin{array}{cc\|c} 2 & 3 & 9 \\ 1 & -1 & -1 \end{array}\right]$$	
b. $\begin{aligned} 2x+4y-\ \ z&=0 \\ x\qquad -4z&=9 \\ y+2z&=6 \end{aligned}$	b. $\begin{aligned} x\qquad -2z&=7 \\ 3x-y-4z&=-1 \\ 11x+y+2z&=2 \end{aligned}$
Solution Since this system has more variables and more equations, it results in a 3 × 4 augmented matrix. $$\left[\begin{array}{ccc\|c} 2 & 4 & -1 & 0 \\ 1 & 0 & -4 & 9 \\ 0 & 1 & 2 & 6 \end{array}\right]$$	
c. $\begin{aligned} y&=2x+1 \\ x+y&=7 \end{aligned}$	c. $\begin{aligned} x+3&=-5y+1 \\ 2x-y&=6 \end{aligned}$
Solution The variables are not on the same side of the equation. To fix this, subtract $2x$ from both sides of the first equation to give the system $$\begin{aligned} -2x+y&=1 \\ x+y&=7 \end{aligned}$$ The corresponding augmented matrix is $$\left[\begin{array}{cc\|c} -2 & 1 & 1 \\ 1 & 1 & 7 \end{array}\right]$$	

Guided Example	Practice
For each of the augmented matrices below, write the corresponding system of equations. a. $\left[\begin{array}{cc\|c} 2 & -1 & 5 \\ 3 & 1 & 4 \end{array}\right]$ **Solution** Assume that the first and second columns correspond t the variables x and y respectively. The corresponding system of linear equations is $$\begin{aligned} 2x - y &= 5 \\ 3x + y &= 4 \end{aligned}$$ b. $\left[\begin{array}{ccc\|c} 1 & 0 & 3 & 2 \\ 0 & 3 & 5 & 2 \\ 0 & 2 & -7 & 10 \end{array}\right]$ **Solution** Assume that the first three columns correspond to x, y, and z. The corresponding system of linear equations is $$\begin{aligned} x \quad + 3z &= 2 \\ 3y + 5z &= 2 \\ 2y - 7z &= 10 \end{aligned}$$	2. For each of the augmented matrices below, write the corresponding system of equations. a. $\left[\begin{array}{cc\|c} -3 & 9 & 1 \\ 2 & -5 & 3 \end{array}\right]$ b. $\left[\begin{array}{ccc\|c} 1 & 2 & -1 & 7 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 3 & 9 \end{array}\right]$

Guided Example	Practice
Using the variables x, y and z, write the solution corresponding to the augmented matrix $$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$ **Solution** The corresponding system of linear equations is $$\begin{aligned} x &= 4 \\ y &= 2 \\ z &= -1 \end{aligned}$$	3. Using the variables x, y and z, write the solution corresponding to the augmented matrix $$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Question 3 - How do you use row operations to determine the reduced row echelon form of a matrix?

Key Terms

Reduced row echelon form

Equivalent matrix

Summary

To solve a system of linear equations using matrices, we follow a strategy

Strategy for Solving a System of Linear Equations with Matrices

1. Determine the augmented matrix that corresponds to the given system of equations.
2. Find the reduced row echelon form of the augmented matrix.
3. Use the reduced row echelon form to obtain the solution of the original system of equations.

A system is in reduced row echelon form if the leading nonzero entry in each row is a one and all entries above and below the leading ones are zeros. To get the matrix into reduced row echelon form, we need to carry out row operations on the augmented matrix. Row operations yield augmented matrices that are equivalent. In other words, they correspond to systems that have the same solutions.

Row Operations on Matrices

Each of the row operations below changes a matrix to an equivalent matrix:

1. Interchange any two rows of a matrix.

2. Multiply a row by a nonzero constant.

3. Replace a row with the sum of a nonzero multiple of one row to a nonzero multiple of another row.

The process of converting an augmented matrix to reduced row echelon form is called Gauss-Jordan Elimination.

Let's look at how we would carry out row operations to solve the system of linear equations,

$$\begin{aligned} x+2y-\ 5z&=-18 \\ 3x+5y+13z&=32 \\ -2x-2y+\ 8z&=26 \end{aligned}$$

Operation	Arithmetic	Corresponding Matrix
		$\left[\begin{array}{ccc\|c} 1 & 2 & -5 & -18 \\ 3 & 5 & 13 & 32 \\ -2 & -2 & 8 & 26 \end{array}\right]$
$-3R_1+R_2 \to R_2$ $2R_1+R_3 \to R_3$	$\begin{array}{cccc} -3 & -6 & 15 & 54 \\ 3 & 5 & 13 & 32 \\ \hline 0 & -1 & 28 & 86 \end{array}$ $\begin{array}{cccc} 2 & 4 & -10 & -36 \\ -2 & -2 & 8 & 26 \\ \hline 0 & 2 & -2 & -10 \end{array}$	$\left[\begin{array}{ccc\|c} 1 & 2 & -5 & -18 \\ 0 & -1 & 28 & 86 \\ 0 & 2 & -2 & -10 \end{array}\right]$
$-1R_2 \to R_2$	$\begin{array}{cccc} 0 & -1 & 28 & 86 \\ & & \times & -1 \\ \hline 0 & 1 & -28 & -86 \end{array}$	$\left[\begin{array}{ccc\|c} 1 & 2 & -5 & -18 \\ 0 & 1 & -28 & -86 \\ 0 & 2 & -2 & -10 \end{array}\right]$
$-2R_2+R_1 \to R_1$	$\begin{array}{cccc} 0 & -2 & 56 & 172 \\ 1 & 2 & -5 & -18 \\ \hline 1 & 0 & 51 & 154 \end{array}$	$\left[\begin{array}{ccc\|c} 1 & 0 & 51 & 154 \\ 0 & 1 & -28 & -86 \\ 0 & 0 & 54 & 162 \end{array}\right]$

$-2R_2 + R_3 \to R_3$	$\begin{array}{cccc} 0 & -2 & 56 & 172 \\ 0 & 2 & -2 & -10 \\ \hline 0 & 0 & 54 & 162 \end{array}$	
$\frac{1}{54}R_3 \to R_3$	$\begin{array}{cccc} 0 & 0 & 54 & 162 \\ & & & \times\frac{1}{54} \\ \hline 0 & 0 & 1 & 3 \end{array}$	$\left[\begin{array}{ccc:c} 1 & 0 & 51 & 154 \\ 0 & 1 & -28 & -86 \\ 0 & 0 & 1 & 3 \end{array}\right]$
$-51R_2 + R_1 \to R_1$ $28R_2 + R_3 \to R_3$	$\begin{array}{cccc} 0 & 0 & -51 & -153 \\ 1 & 0 & 51 & 154 \\ \hline 1 & 0 & 0 & 1 \end{array}$ $\begin{array}{cccc} 0 & 0 & 28 & 84 \\ 0 & 1 & -28 & -86 \\ \hline 0 & 1 & 0 & -2 \end{array}$	$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

This gives the solution, $x = 1$, $y = -2$, and $z = 3$.

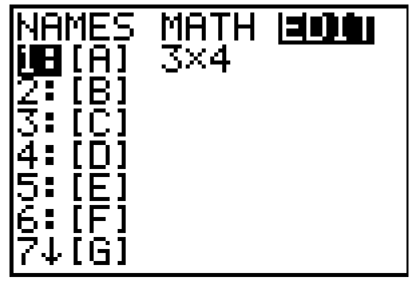
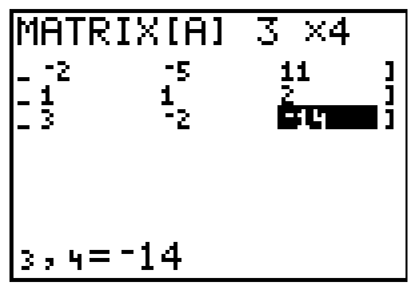
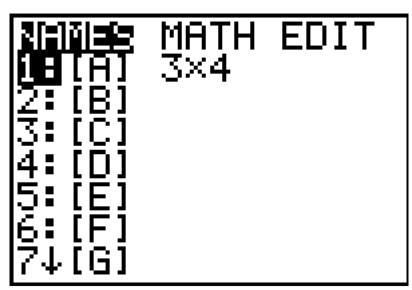
A graphing calculator also contains a command that will put a matrix into reduced row echelon form. For example, suppose we want to utilize your graphing calculator to solve the system

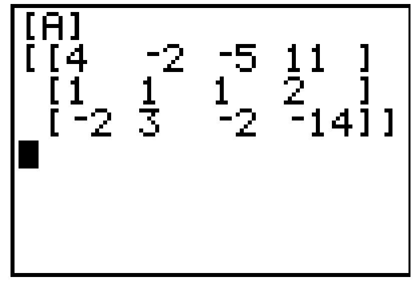
$$\begin{aligned} 4x - 2y - 5z &= 11 \\ x + y + z &= 2 \\ -2x + 3y - 2z &= -14 \end{aligned}$$

Start by converting this system to an augmented matrix,

$$\left[\begin{array}{ccc:c} 4 & -2 & -5 & 11 \\ 1 & 1 & 1 & 2 \\ -2 & 3 & -2 & -14 \end{array}\right]$$

Your calculator can put a matrix into reduced row echelon form using the rref command.

Enter the Matrix 1. Press ψ □ to access the MATRIX menu. 2. Use ~ to go to EDIT. 3. Press ℵ or move the cursor to 1: [A] and press ⊆. Note that if you used this matrix name before, it will have a dimension next to it.	
4. Enter the dimension of matrix A as 3 x 4. 5. Enter the values into the matrix as shown. Press ⊆ after each entry. Note that the position is given at the bottom of the screen as 3,1=1 etc. This matrix will need two screens. Use ~ to see last column and to enter. 6. Press ψ ζ to QUIT and return to the home screen.	
View the Matrix on the Home Screen 7. Press ψ □ to access the MATRIX menu. You are in the NAMES menu.	

8. Move the cursor to 1: [A] and press ⊆. This will put [A] on the Home screen. 9. Press ⊆ to view the matrix on the home screen. You may need to use the right arrow to scroll through the entire matrix.	

<u>Notes</u>

Guided Example

Carry out the row operation: Replace R_1 with $\frac{1}{2}R_1$.

$$\left[\begin{array}{ccc|c} 2 & 4 & -10 & 2 \\ 4 & 2 & -3 & 3 \\ 1 & -3 & 4 & 2 \end{array}\right]$$

Solution Multiply each entry in the first row by $\frac{1}{2}$:

$$\begin{array}{cccc} 2 & 4 & -10 & \vdots\ 2 \\ & & \times & \frac{1}{2} \\ \hline 1 & 2 & -5 & \vdots\ 1 \end{array}$$

And replace the first row with this product,

$$\left[\begin{array}{ccc|c} 1 & 2 & -5 & 1 \\ 4 & 2 & -3 & 3 \\ 1 & -3 & 4 & 2 \end{array}\right]$$

Practice

1. Carry out the row operation: Replace R_2 with $\frac{1}{4}R_2$.

$$\left[\begin{array}{ccc|c} 2 & 4 & -10 & 2 \\ 4 & 2 & -3 & 3 \\ 1 & -3 & 4 & 2 \end{array}\right]$$

Guided Example

Carry out the row operation: Replace R_3 with $-3R_1 + R_3$.

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 1 \\ 0 & 2 & 2 & -1 \\ 3 & 1 & 2 & -1 \end{array}\right]$$

Solution Start by working out the row operation

$$\begin{array}{rcccc} -3R_1: & -3 & -9 & 3 & \vdots\ -3 \\ +R_3: & 3 & 1 & 2 & \vdots\ -1 \\ \hline & 0 & -8 & 5 & \vdots\ -4 \end{array}$$

Practice

2. Carry out the row operation.

a. Replace R_2 with $-2R_1 + R_2$.

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 2 & 4 & 2 & 10 \\ 5 & 4 & 2 & 1 \end{array}\right]$$

And then replace the third row with this new row $$\left[\begin{array}{ccc\|c} 1 & 3 & -1 & 1 \\ 0 & 2 & 2 & -1 \\ 0 & -8 & 5 & -4 \end{array}\right]$$	b. Starting from your answer for part a, replace R_3 with $-5R_1 + R_3$.

Guided Example

Practice

Solve the system of equations below with Gauss-Jordan elimination. $$\begin{aligned} x + \quad y &= 10 \\ 0.08x + 0.12y &= 0.84 \end{aligned}$$ **Solution** Rewrite the system in an augmented matrix to get $$\left[\begin{array}{cc\|c} 1 & 1 & 10 \\ .08 & .12 & .84 \end{array}\right]$$ The leading entry in the first row is already a 1, so change the first entry, second row to a zero: $$-0.08R_1 + R_2 \to R_2 \quad \left[\begin{array}{cc\|c} 1 & 1 & 10 \\ 0 & .04 & .04 \end{array}\right]$$	3. Solve the system of equations below with Gauss-Jordan elimination. $$\begin{aligned} 2x + 5y &= -6 \\ -3x + y &= -8 \end{aligned}$$

To put a 1 in place of the leading entry in the second row: $\tfrac{1}{.04}R_2 \to R_2 \quad \left[\begin{array}{cc:c} 1 & 1 & 10 \\ 0 & 1 & 1 \end{array}\right]$ Finally, change the 1 in the first row, second column to a zero: $-1R_2 + R_1 \to R_1 \quad \left[\begin{array}{cc:c} 1 & 0 & 9 \\ 0 & 1 & 1 \end{array}\right]$ This gives the solution $(x, y) = (9,1)$.	

Question 4 - Do all systems of linear equations have unique solutions?

Key Terms

Summary

We can use Gauss-Jordan Elimination to solve a system when it does not have a solution or has many solutions. Start by carrying out the process of placing 1's and 0's in the matrix using row operations. Once the matrix is in reduced row echelon form, convert the matrix back to an equation. Once we have the equations, you would follow the same procedure that we used in Section 2.3 for inconsistent and dependent systems.

For instance, suppose a system of linear equations results in the reduced row echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 10 \\ 0 & 1 & 2 & 12 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

The last row of the matrix converts to the equation 0 = 1. Since this is a contradiction, the original system of equations has no solutions.

Suppose a system of linear equations results in the reduced row echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last row of the augmented matrix corresponds to the identity 0 = 0. In this situation, we rewrite the first two rows of the matrix as equations,

$$\begin{aligned} x + z &= 5 \\ y - z &= 6 \end{aligned}$$

And solve for x and y to get

$$\begin{aligned} x &= -z + 5 \\ y &= z + 6 \end{aligned}$$

This corresponds to the ordered triple $(-z+5, z+6, z)$.

Notes

Guided Example

Solve the system of equations below with Gauss-Jordan elimination.

$$\begin{aligned} x + \quad y + \quad z &= 120 \\ 0.04x + 0.06y + 0.1z &= 6 \end{aligned}$$

Solution The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 120 \\ 0.04 & 0.06 & 0.1 & 6 \end{array}\right]$$

Follow the steps to put the augmented matrix in reduced row echelon form:

$$-0.04R_1 + R_2 \to R_2 \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 120 \\ 0 & 0.02 & 0.06 & 1.2 \end{array}\right]$$

$$\tfrac{1}{0.02} R_2 \to R_2 \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 120 \\ 0 & 1 & 3 & 60 \end{array}\right]$$

$$-1R_2 + R_1 \to R_1 \quad \left[\begin{array}{ccc|c} 1 & 0 & -2 & 60 \\ 0 & 1 & 3 & 60 \end{array}\right]$$

The augmented matrix is now in reduced row echelon form. Converting it back to equations gives

$$\begin{aligned} x - 2y &= 60 \\ y + 3y &= 60 \end{aligned}$$

Solving for x and y yields

$$\begin{aligned} x &= 2z + 60 \\ y &= -3z + 60 \end{aligned}$$

Which corresponds to the ordered triple $(2z + 60, -3z + 60, z)$.

Practice

1. Solve the system of equations below with Gauss-Jordan elimination.

$$\begin{aligned} x + \quad y - \quad z &= 10 \\ 2x - \quad y - \quad z &= 3 \\ -3x - 3y + 3z &= -30 \end{aligned}$$

Question 5 - How do you mix different grades of ethanol to create a new grade of ethanol?

Key Terms

Summary

In this question, we look at examples of system of equations that come from real problems. In each case, we follow several basic steps to write out the system of equations. Once we have the system, convert the augmented matrix to reduced row echelon form using the rref command on a graphing calculator or Gauss-Jordan Elimination.

The basic steps for solving an application problem are

1. Identify what you are looking for from the problem statement. Assign variables to what you are looking for and describe the variables in detail.
2. Identify key information in the problem statement that relates the variables.
3. Write out equations that correspond to the key information.
4. Solve the system with an appropriate technique.

Let's demonstrate these steps with an example.

> The Riddler Rent-A-Truck company plans to spend $7 million on 200 new vehicles. Each commercial van will cost $35,000, each small truck $30,000, and each large truck $50,000. They need twice as many vans as small trucks. How many of each vehicle can they buy?

Start by defining the variables:

V: number of commercial vans to buy

S: number of small trucks to buy

L: number of large trucks to buy

Now let's look at the key information and the corresponding equations:

buy 200 new vehicles $\rightarrow \quad V + S + L = 200$

spend 7 million $\rightarrow \quad 35000V + 30000S + 50000L = 7000000$

need twice as many vans as small trucks $\rightarrow \quad V = 2S$

The system we need to solve is

$$\begin{aligned} V + S + L &= 200 \\ 35000V + 30000S + 50000L &= 7000000 \\ V &= 2S \end{aligned}$$

Rewriting in the proper form, we get

$$\begin{aligned} V + S + L &= 200 \\ 35000V + 30000S + 50000L &= 7000000 \\ V - 2S &= 0 \end{aligned}$$

The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 200 \\ 35000 & 30000 & 50000 & 7000000 \\ 1 & -2 & 0 & 0 \end{array}\right]$$

Use a graphing calculator or Gauss-Jordan Elimination to get the reduced row echelon form,

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 120 \\ 0 & 1 & 0 & 60 \\ 0 & 0 & 1 & 20 \end{array}\right]$$

or 120 vans, 60 small trucks and 20 large trucks. Note that these numbers match the key information in the problem:

buy 200 new vehicles $\rightarrow \quad 120 + 60 + 20 = 200$

spend 7 million $\rightarrow \quad 35000(120) + 30000(60) + 50000(20) = 7000000$

need twice as many vans as small trucks→ 120 is twice as big as 60

Notes

Guided Example	Practice
In December of 2014, Sony released the movie *The Interview* online after threats to theaters cancelled the debut in theaters. As originally reported in Wall Street Journal, Sony reported sales of $31 million from the sales and rentals of *The Interview*. They sold the movies online for $15 and rented through various sites for $6. If there were 4.3 million transactions, how many of the transaction were sales of the movie and how many of the transactions were rentals? **Solution** Let's start by defining two variables to what we are looking for: S: number of sales transactions for *The Interview* in millions R: number of rental transactions for *The Interview* in millions These variables also relate to the total sales as well as the total number of transactions. Since there were a total of 4.3 million transactions, $$S + R = 4.3$$ Each sale of the movie yields $15 in sales and each rental results in $6 in sales. Thus, the total sales yields $$15S + 6R = 31$$ Putting these equations together gives the system of linear equations, $$\begin{aligned} S + \quad R &= 4.3 \\ 15S + 6R &= 31 \end{aligned}$$ Using an augmented matrix, we can write the system as	1. Katherine Chong invests $10,000 received from her grandmother in three ways. With one part, she buys US savings bonds at an interest rate of 2.5% per year. She uses the second part, which amounts to twice the first, to buy mutual funds that offer a return of 6% per year. She puts the rest of the money into a money market account paying 4.5% annual interest. The first year her investments bring a return of $470. How much did she invest in each way?

$$\left[\begin{array}{cc|c} 1 & 1 & 4.3 \\ 15 & 6 & 31 \end{array}\right]$$

Well put this into reduced row echelon form using row operations. The entry in the first row, first column is a 1, so we put a 0 in the second row, first column:

$$-15R_1 + R_2 \to R_2 \quad \left[\begin{array}{cc|c} 1 & 1 & 4.3 \\ 0 & -9 & -33.5 \end{array}\right]$$

Now put a 1 in the second row, second column:

$$-\tfrac{1}{9}R_2 \to R_2 \quad \left[\begin{array}{cc|c} 1 & 1 & 4.3 \\ 0 & 1 & \frac{33.5}{9} \end{array}\right]$$

Finally, put a zero in the first row, second column:

$$-R_2 + R_1 \to R_1 \quad \left[\begin{array}{cc|c} 1 & 0 & \frac{26}{45} \\ 0 & 1 & \frac{33.5}{9} \end{array}\right]$$

This gives the solution $S = \frac{26}{45} \approx 0.58$ and $R = \frac{33.5}{9} \approx 3.72$. Remember that each variable is in millions so there were approximately 0.58 million sale transactions and 3.72 million rental transactions. Based on the critical reviews of the movies as well as the movie *Pineapple Express*, the large number of rentals was justified.

Chapter 2 Solutions

Section 2.1

Question 1 1) a. Not a solution, b. Solution

2) (-1, -1)

3) (2, -1)

Question 2 1) a. $168, b. 2400 DVD players

2) Demand: $p = -0.05q + 100$, Supply: $p = 0.06q + 6.5$ so (850, 57.50) is the market equilibrium. At a price of $57.50, the quantity demanded and supplied are both 850 units.

3) 500 units

Section 2.2

Question 1 1) (-3, 9), 2) (2,1), 3) (1, -4), 4) (1, 4) 5) when price is $100 and quantity is 50,000.

Section 2.3

Question 1 1) no solution, 2) $\left(\frac{3}{2}y + \frac{5}{2}, y\right)$ 3) 22 noninteger solutions from $z = 0$ to $z = 21$.

4) $\left(\frac{5}{3}z + \frac{11}{3}, -\frac{7}{3}z - \frac{4}{3}, z\right)$

Question 2 1) (5, 3)

Question 3 1) $26000/7 in US bonds, $44000/7 in mutual funds, $0 in money market.

Section 2.4

Question 1 1) a. 3 × 3, b. -3, c. 0

Question 2 1) a. $\left[\begin{array}{cc|c} -5 & 2 & 12 \\ 3 & -4 & 1 \end{array}\right]$, b. $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 7 \\ 3 & -1 & -4 & -1 \\ 11 & 1 & 2 & 2 \end{array}\right]$, c. $\left[\begin{array}{cc|c} 1 & 5 & -2 \\ 2 & -1 & 6 \end{array}\right]$

2) a. $\begin{aligned} -3x+9y&=1 \\ 2x-5y&=3 \end{aligned}$, b. $\begin{aligned} x+2y-z&=7 \\ y+4z&=-1 \\ 3z&=9 \end{aligned}$

3) (-1, 5, 2)

Question 3 1) $\left[\begin{array}{ccc|c} 2 & 4 & -10 & 2 \\ 1 & \frac{1}{2} & -\frac{3}{4} & \frac{3}{4} \\ 1 & -3 & 4 & 2 \end{array}\right]$

2) a. $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 4 & 4 & 6 \\ 5 & 4 & 2 & 1 \end{array}\right]$, b. $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 4 & 4 & 6 \\ 0 & 4 & 7 & 9 \end{array}\right]$

3) (2, -2)

Question 4 1) $\left(\frac{2}{3}z+\frac{13}{3}, \frac{1}{3}z+\frac{17}{3}, z\right)$

Question 5 1) \$2000 in US Savings Bonds, \$4000 in mutual funds and \$4000 in money market accounts