Section 4.1 Solving Systems of Linear Inequalities

Question 1 – How do you graph a linear inequality?

Question 2 – How do you graph a system of linear inequalities?

Question 1 – How do you graph a linear inequality?

<u>Key Terms</u>

Linear inequality

<u>Summary</u>

A linear inequality is a linear equation where the equal sign has been changed an inequality like $<$, $>$, $\leq$, or $\geq$. The solution set to a linear inequality is a region of the xy plane that is bordered by a line. If the inequality includes an equal sign ($\leq$ or $\geq$), then the border is drawn with a solid line. The solid line indicates that the border is a part of the solution set to the inequality. If the inequality does not include an equal sign ($<$ or $>$), then the border is drawn with a dashed line. The dashed line indicates that the border is not a part of the solution set to the inequality.

To graph the solution set to an inequality,

1. Identify the independent and dependent variables. Begin the graph of the solution set by labeling the independent variable on the horizontal axis and the dependent variable on the vertical axis.
2. Change the inequality to an equation by replacing the inequality with an equal sign.
3. Graph the equation using the intercepts or another convenient method. If the inequality is a strict inequality, like $<$ or $>$, graph the line with a dashed line. If the inequality includes an equal sign, like $<$ or $>$, graph the line as a solid line.
4. Pick a test point to substitute into the inequality. Test points that include zeros are easiest to work with. This test point must not be a point on the line.
5. If substituting the test point into the inequality makes it true, shade the side of the line containing the test point. If substituting the test point into the inequality makes it false, shade the side of the line that does not contain the test point.

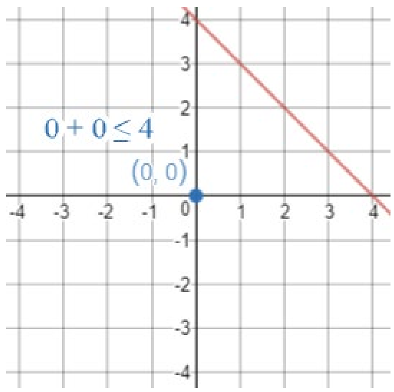
<u>Notes</u>

Guided Example	Practice

Graph the inequality on a plane.

$$x + y \leq 4$$

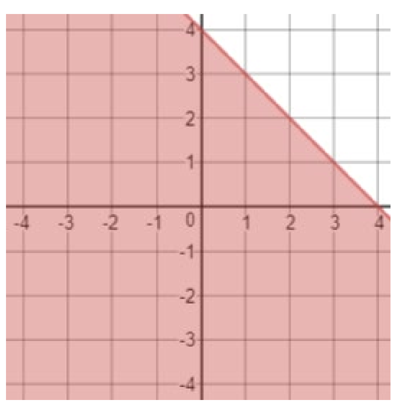
Solution Start by choosing x to be the independent variable and y the dependent variable. The equation of the border of the solution set is $x + y = 4$. We can graph this equation using intercepts or by rewriting the equation as $y = -x + 4$. This results in the graph below.



Since the inequality includes an equal sign, draw the line as a solid line. To see which side of the line to shade, test the point (0,0) in the inequality:

$$0 + 0 \leq 4 \quad \text{TRUE}$$

Since the inequality is true, all points on that side of the line satisfy the inequality and the solution set is



1. Graph the inequality on a plane.

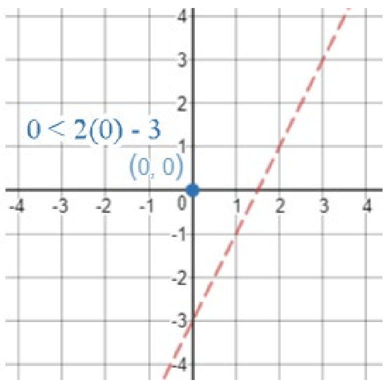
$$x - y \geq 1$$

Guided Example

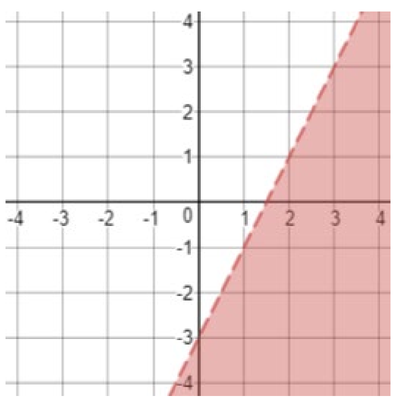
Graph the inequality.

$$y < 2x - 3$$

Solution The equation of the border is $y = 2x - 3$. Graph the border with a dashed line and test the point (0, 0) in the inequality.



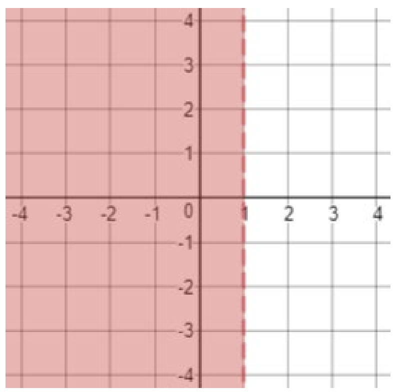
Since 0 < -3 is false, the solution set is on the other side of the line.



Practice

2. Graph the inequality.

$$y < -2x + 3$$

Guided Example	Practice
Graph the inequality on a plane. $$x < 1$$ **Solution** The graph of $x = 1$ is a vertical line. When we test the point (0, 0) in the inequality, we get $0 < 1$ which is true. Shade the side of the line that include this point to get the solution set.  Notice that the border is drawn with a dashed line since the inequality does not include and equals sign.	3. Graph the inequality on a plane. $$y > 1$$

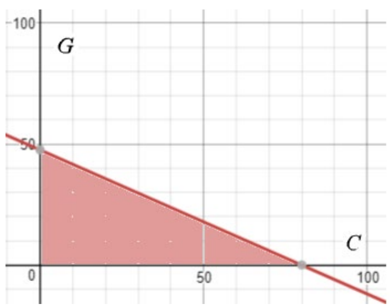
Guided Example	Practice
Hops costs for a brewer are \$25 per pound for Citra hops and \$42 per pound for Galena hops. How many pounds of each hops should the brewer use if she wants to spend no more than \$2000 on hops? Express your answer as a linear inequality with appropriate nonnegative restrictions and draw its graph. **Solution** Start by defining the variable C for the number of pounds of Citra hops and G for the number of pounds of Galena hops. Either variable may be chosen as the independent variable. For this example, we'll choose C as the independent variable. If Citra hops cost \$25 per pound, then C pounds will cost $25C$. Similarly, G pounds of Galena	4. Ingredient costs for a pet food manufacturer are \$5 per pound for vegetables and \$8 per pound for meat. How many pounds of each ingredient should the manufacturer use if she wants to spend no more than \$4000 on ingredients? Express your answer as a linear inequality with appropriate nonnegative restrictions and draw its graph.

hops will cost 42G. Since the total cost must be no more than \$2000,

$$25C + 42G \le 2000$$

Since (0, 0) makes this inequality true, the solution to this inequality is

However, the variables in this problem represent pounds of hops and can't be negative. This means the solution must lie only in the first quadrant.



Question 2 – How do you graph a system of linear inequalities?

Key Terms

System of linear inequalities | Bounded region

Unbounded region | Feasible region

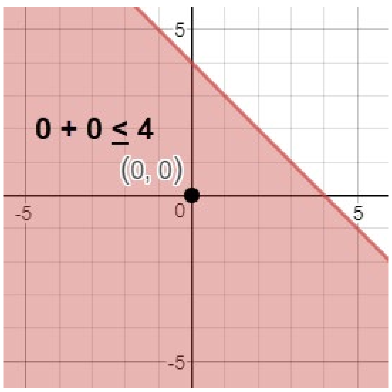
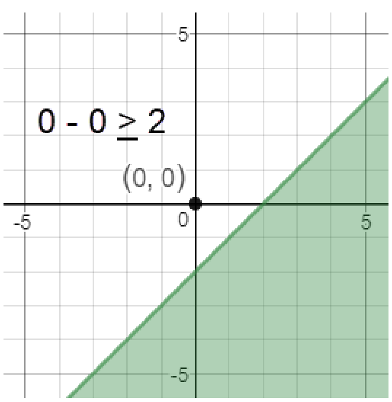
Summary

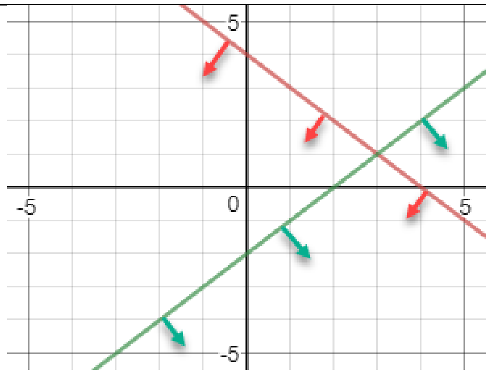
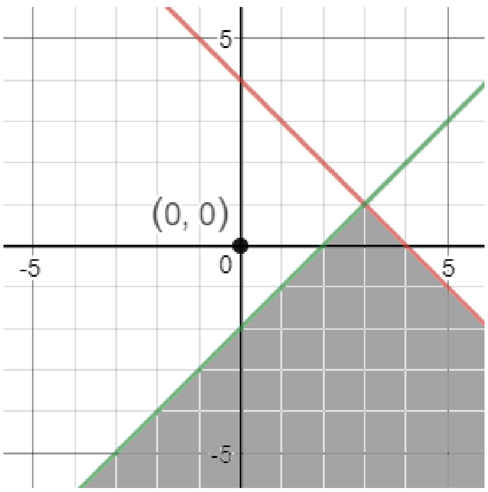
A system of linear inequalities is a group of more than one inequality. To graph a system of linear inequalities,

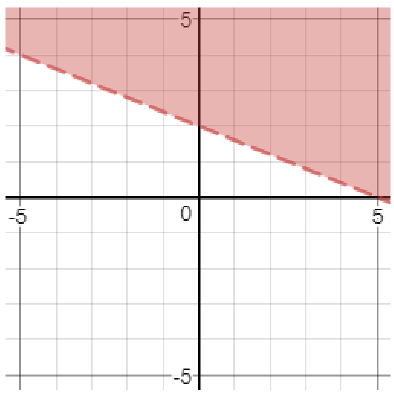
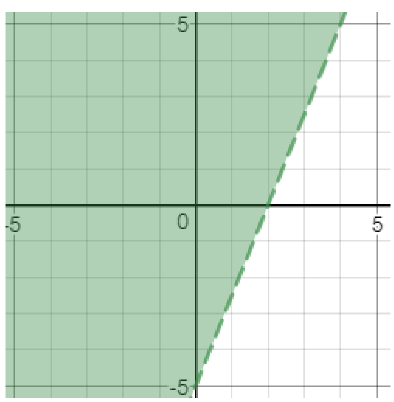
1. Graph the corresponding linear equation for each of the linear inequalities. If the inequality includes an equal sign, graph the equation with a solid line. If the inequality does not include an equals, graph the equation with a dashed line.
2. For each inequality, use a test point to determine which side of the line is in the solution set. Instead of using shading to indicate the solution, use arrows along the line pointing in the direction of the solution.
3. The solution to the system of linear inequalities is all areas on the graph that are in the solution of all of the inequalities. Shade any areas on the graph that the arrows you drew indicate are in common.

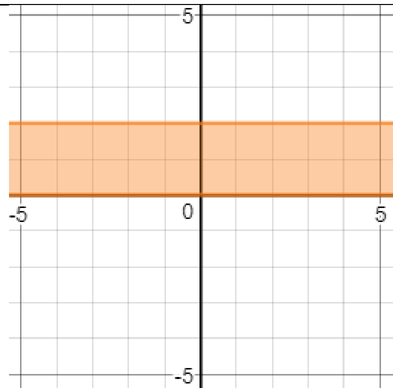
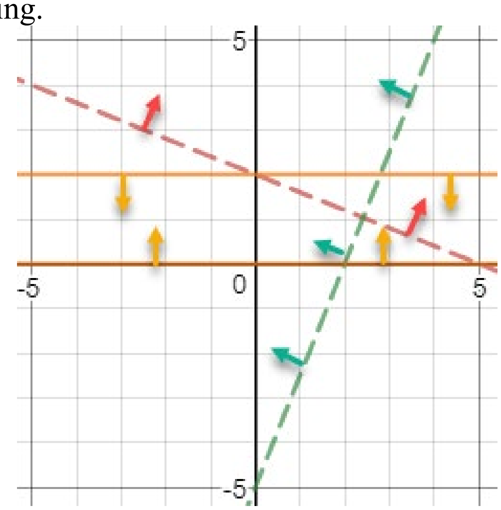
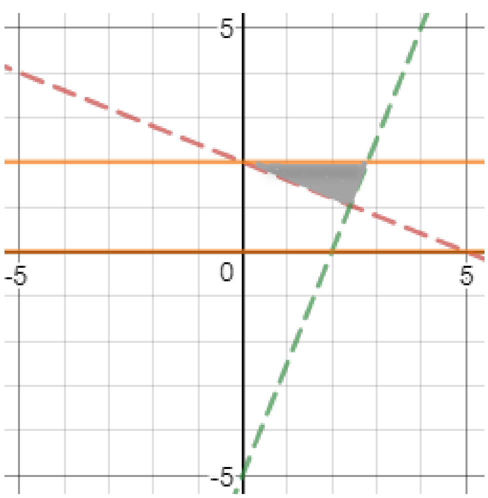
The solution set is often called a feasible region. The feasible region is bounded if it is surrounded by borders on all sides. If the feasible region extends infinitely far in any direction, it is unbounded.

Notes

<u>Guided Example</u>	<u>Practice</u>
Graph the feasible region for the following system of inequalities. Tell whether the region is bounded or unbounded. $x+y \le 4$ $x-y \ge 2$ **Solution** Start by graphing the border of the first inequality $x+y=4$. Then test the point (0, 0) in the inequality.  Since the inequality is true, shade the side of the line that (0, 0) lies in. Similarly, shade the line $x-y=6$ and test (0, 0).  Since the inequality is false, shade the opposite side of the line. Using arrows to indicate the shading the region looks like this,	1. Graph the feasible region for the following system of inequalities. Tell whether the region is bounded or unbounded. $x+y \ge 2$ $x-y \le 4$

 These individual solution sets overlap in the solution set of the system of inequalities. 	

<u>Guided Example</u>	<u>Practice</u>
Graph the solution of the system of linear inequalities. $2x+5y>10$ $5x-2y<10$ $0 \le y \le 2$ **Solution** Graph the first inequality $2x+5y>10$.  The second inequality $5x-2y<10$ results in the graph below.  And finally $0 \le y \le 2$.	2. Graph the solution of the system of linear inequalities. $6x+3y>18$ $3x-6y<18$ $0 \le y \le 1$

 Let's put these together with arrows indicating the shading.  These solution sets overlap in the triangular solution set to the system. 	

Guided Example

A pet warehouse is planning to make a package of dog treats containing vegetables and meat. Each ounce of vegetables will supply 1 unit of protein, 2 units of carbohydrates, and 0.25 unit of fat. Each ounce of meat will supply 1 unit of protein, 1 unit of carbohydrates, and 1 unit of fat. Every package must provide at least 55 units of protein, at least 12 units of carbohydrates, and no more than 88 units of fat. Let x equal the ounces of vegetable and y equal the ounces of meat to be used in each package.

a. Write a system of inequalities to express the conditions of the problem.

Solution The variables are defined in the problem statement. To get started on the inequalities, look for any totals in the problem statement like, "Every package must provide at least 55 units of protein". Since an ounce of vegetables provide 1 unit of protein, x ounces of vegetables will provide $1x$ units of vegetables. The protein provided by y ounces of meat is $1y$. This means that we can write a protein constraint

$$x + y \geq 55$$

The constraint for carbohydrates is

$$2x + y \geq 12$$

and the constraint for fat is

$$0.25x + y \leq 88$$

Notice that this inequality is less than or equal to since the problem statement said "no more than 88 units of fat ". In addition to these constraints, we know that each variable should be nonnegative,

$$x \geq 0 \quad y \geq 0$$

Putting these inequalities together gives the system of inequalities,

Practice

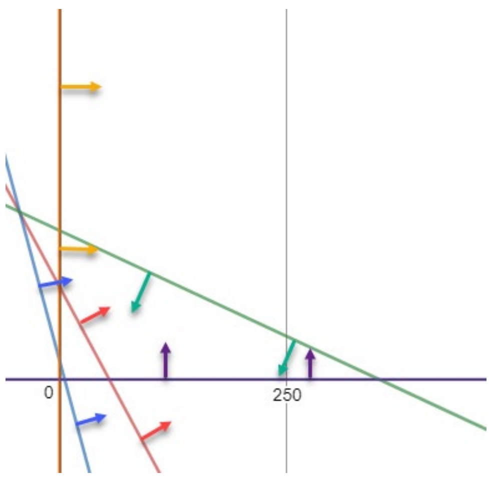
3. An investor wishes to invest no more than $10,000 in two stocks. The first stock, Aramid Inc., has a dividend of 1.5%. The second stock, Blue Deuce Insurance, has a dividend of 2%. The investor wishes a total dividend of more than $144 from these stocks. In addition, the investor wants the amount invested in Aramid to be greater than the amount invested in Blue Deuce. . Let x equal the amount invested in Aramid and y equal the amount invested in Blue Deuce.

a. Write a system of inequalities to express the conditions of the problem.

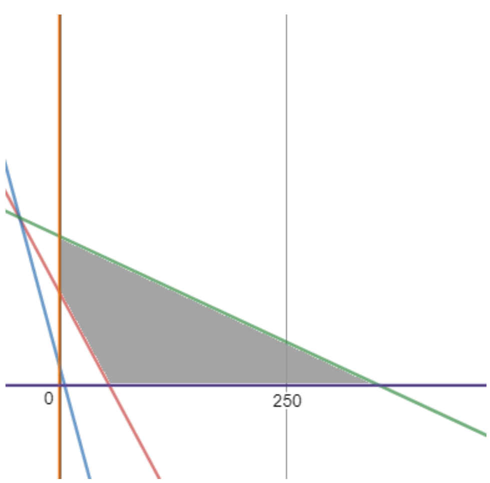
$$x + y \geq 55$$
$$2x + y \geq 12$$
$$0.25x + y \leq 88$$
$$x \geq 0 \quad y \geq 0$$

b. Graph the feasible region of the system.

Solution Draw a line for each border and use arrows to indicate where the solution set for that inequality is. This makes it easier to see where the individual solution sets cross.



The solution to the system of inequalities is the gray feasible region below.



b. Graph the feasible region of the system.

Section 4.2 Graphical Linear Programming

Question 1 – What is a linear programming problem?

Question 2 – How do you solve a linear programming problem with a graph?

Question 1 – What is a linear programming problem?

Key Terms

Linear programming problem	Objective function
Decision variables	Optimal solution

Summary

A linear programming problem consists of an objective function and a system of inequalities that defines acceptable values for the decision variables x_1 through x_n. The objective function is a linear function of the variables x_1 through x_n and is preceded by the word "maximize" or "minimize".

A linear programming problem has the form

Maximize or Minimize $z = a_1x_1 + a_2x_2 + \cdots + a_nx_n$

subject to constraints of the form

$b_1x_1 + b_2x_2 + \cdots + b_nx_n \le c$ or $b_1x_1 + b_2x_2 + \cdots + b_nx_n \ge c$

and $x_1 \ge 0,\ x_2 \ge 0,\ \ldots,\ x_n \ge 0$.

In a linear programming problem, the constants $a_1, \ldots, a_n$, $b_1, \ldots, b_n$, and c are real numbers. There are usually several inequalities in the system of inequalities. These inequalities constrain the values that the decision variables may take and are typically called constraints. The inequalities that requires the variables to be nonnegative are called nonnegativity constraints.

The values of the decision variables that optimize (maximize or minimize) the value of the objective function are called the optimal solution of the linear programming problem.

Notes

Guided Example	Practice
Suppose you are given the linear programming problem Maximize $z = 10x_1 + 8x_2$ subject to $x_1 + x_2 \le 50$ $6x_1 + 12x_2 \le 480$ $x_1 \ge 0,\ x_2 \ge 0$ a. Identify the objective function. **Solution** The objective function is the function that is to be maximized or minimized, $z = 10x_1 + 8x_2$ b. Identify the constraints. **Solution** The constraints are formed by the system of inequalities, $x_1 + x_2 \le 50$ $6x_1 + 12x_2 \le 480$ $x_1 \ge 0,\ x_2 \ge 0$ c. Identify the nonnegativity constraints. **Solution** The nonnegativity constraints are the constraints that cause us to consider solutions that are not negative, $x_1 \ge 0,\ x_2 \ge 0$	1. Suppose you are given the linear programming problem Maximize $z = 3x_1 + 4x_2$ subject to $x_1 + x_2 \le 40$ $x_1 + 2x_2 \le 60$ $x_1 \ge 0,\ x_2 \ge 0$ a. Identify the objective function. b. Identify the constraints. c. Identify the nonnegativity constraints.

Question 2 – How do you solve a linear programming problem with a graph?

Key Terms

Feasible region Corner point

Summary

For a linear programming problem in two decision variables, the system of inequalities defines a region from which the optimal solution may come from. This region is called the feasible region for the linear programming problem. This region consists of line segments that meet at corner points. The optimal solution to a linear programming problem occurs at a corner point to the feasible region or along a line connecting two adjacent corner points of the feasible region.

If a feasible region is bounded, there will always be an optimal solution. Unbounded feasible regions may or may not have an optimal solution.

We can use this insight to develop the following strategy for solving linear programming problems with two decision variables.

1. Graph the feasible region using the system of inequalities in the linear programming problem.

2. Find the corner points of the feasible region.

3. At each corner point, find the value of the objective function.

By examining the value of the objective function, we can find the maximum or minimum values. If the feasible region is bounded, the maximum and minimum values of the objective function will occur at one or more of the corner points. If two adjacent corner points lead to same maximum (or minimum) value, then the maximum (or minimum) value also occurs at all points on the line connecting the adjacent corner points.

Unbounded feasible regions may or may not have optimal values. However, if the feasible region is in the first quadrant and the coefficients of the objective function are positive, then there is a minimum value at one or more of the corner points. There is no maximum value in this situation. Like a bounded region, if the minimum occurs at two adjacent corner points, it also occurs on the line connecting the adjacent corner points.

Notes

Guided Example

Suppose you are given the linear programming problem

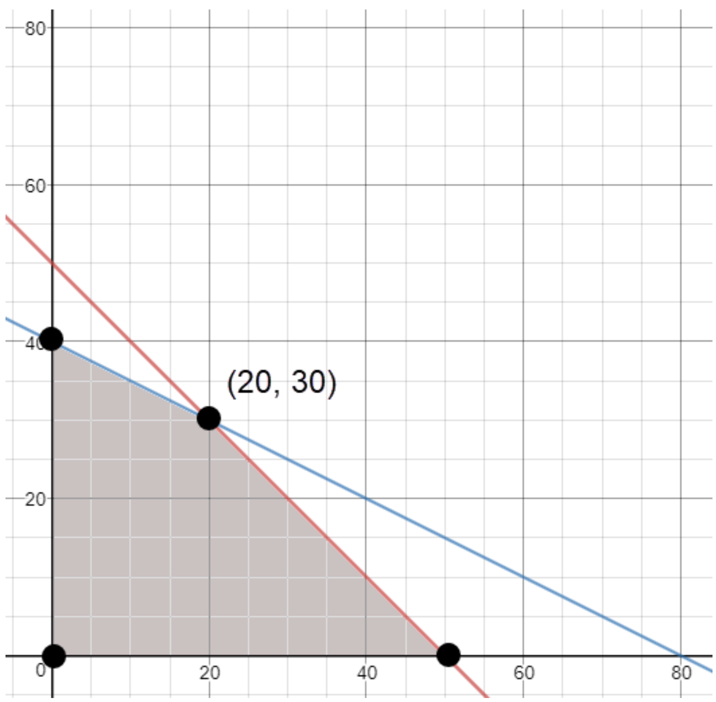
$$\text{Maximize } z = 10x_1 + 8x_2$$

subject to

$$x_1 + x_2 \le 50$$
$$6x_1 + 12x_2 \le 480$$
$$x_1 \ge 0,\ x_2 \ge 0$$

Find the values of x_1 and x_2 that optimize the objective function.

Solution Graph the system of inequalities by graphing the borders, $x_1 + x_2 = 50$ and $6x_1 + 12x_2 = 480$.



The intercepts of these equations give most of the corner points. The other corner points is found by solving the system of equations,

$$x_1 + x_2 = 50$$
$$6x_1 + 12x_2 = 480$$

Solving with substitution or elimination yields the corner point $(x_1, x_2) = (20, 30)$.

To find the corner points with the largest value of *z*, put each corner point in the objective function.

Corner Point (x_1, x_2)	$z = 10x_1 + 8x_2$
(0, 0)	0
(50, 0)	500
(0, 40)	320
(20, 30)	540

The maximum value of z is 540 and occurs when $x_1 = 20$ and $x_2 = 30$.

Practice

1. Suppose you are given the linear programming problem

Maximize $z = 3x_1 + 4x_2$

subject to

$$x_1 + x_2 \le 40$$

$$x_1 + 2x_2 \le 60$$

$$x_1 \ge 0,\ x_2 \ge 0$$

Find the values of x_1 and x_2 that optimize the objective function.

Guided Example

Suppose you are given the linear programming problem

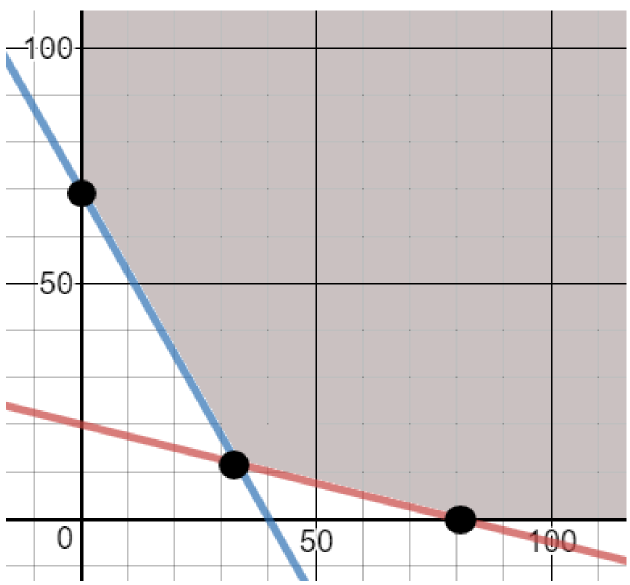
$$\text{Minimize } w = \tfrac{7}{4}y_1 + y_2$$

subject to

$$y_1 + 4y_2 \geq 80$$
$$7y_1 + 4y_2 \geq 280$$
$$y_1 \geq 0,\ y_2 \geq 0$$

Find the values of y_1 and y_2 that optimize the objective function.

Solution Graph the system of inequalities by graphing the borders, $y_1 + 4y_2 = 80$ and $7y_1 + 4y_2 = 280$.



The intercepts of these equations give the corner points at (80, 0) and (0, 70). The other corner points is found by solving the system of equations,

$$y_1 + 4y_2 = 80$$
$$7y_1 + 4y_2 = 280$$

Solving with substitution or elimination yields the corner point $(y_1, y_2) = (\tfrac{100}{3}, \tfrac{35}{3})$.

To find the corner points with the largest value of *w*, put each corner point in the objective function.

Corner Point (y_1, y_2)	$w = \frac{7}{4}y_1 + y_2$
(0, 0)	0
(80, 0)	140
(0, 70)	70
$\left(\frac{100}{3}, \frac{35}{3}\right)$	70

The minimum value of *w* is 70 and occurs at two corner points. This means the minimum is attained at any point on the line connecting these points. So, the minimum is 70 and occurs at any point along the line connecting (0, 70) and $\left(\frac{100}{3}, \frac{35}{3}\right)$.

<u>Practice</u>

2. Suppose you are given the linear programming problem

$$\text{Minimize } w = 2y_1 + y_2$$

subject to

$$y_1 + y_2 \geq 1$$

$$2y_1 + 4y_2 \geq 3$$

$$y_1 \geq 0,\ y_2 \geq 0$$

Find the values of y_1 and y_2 that optimize the objective function.

Section 4.3 The Simplex Method and the Standard Maximization Problem

Question 1 – What is a standard maximization problem?

Question 2 – What are slack variables?

Question 3 - How do you find a basic feasible solution?

Question 4 - How do you get the optimal solution to a standard maximization problem with the Simplex Method?

Question 5 - How do you find the optimal solution for an application?

Question 1 – What is a standard maximization problem?

Key Terms

Standard maximization problem

Summary

A standard maximization problem is a type of linear programming problem in which the objective function is to be maximized and has the form

$$z = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

where $a_1, \ldots, a_n$ are real numbers and $x_1, \ldots, x_n$ are decision variables. The decision variables must represent non-negative values. The other constraints for the standard maximization problem have the form

$$b_1x_1 + b_2x_2 + \cdots + b_nx_n \le c$$

where $b_1, \ldots, b_n$ and c are real numbers and $c \ge 0$.

The variables may have different names, but in standard maximization problems four elements must be present:

1. The objective function is maximized.
2. The objective function must be linear.
3. The constraints are linear where the variables are less than or equal to a nonnegative constant.
4. The decision variables must be nonnegative.

The objective function is maximized

$$\boxed{\text{Maximize}} \quad \boxed{P = 100x_1 + 80x_2}$$

The objective function has the form $z = a_1x_1 + a_2x_2$

subject to

$$\boxed{\begin{aligned} x_1 + x_2 &\le 50{,}000 \\ 69.75x_1 + 85.25x_2 &\le 4{,}000{,}000 \\ 23.8x_1 + 10.85x_2 &\le 1{,}000{,}000 \end{aligned}}$$

All constraints have the form $b_1x_1 + b_2x_2 \le c$ where c is nonnegative.

$$\boxed{x_1 \ge 0,\ x_2 \ge 0}$$

Decision variables are nonnegative

Notes

<u>Guided Example</u> <u>Practice</u>

Is the linear programming problem

$$\text{Maximize } z = 5x_1 + 6x_2$$

subject to

$$2x_1 + x_2 \le 4$$

$$x_1 + 2x_2 \le 4$$

$$x_1 \ge 0,\ x_2 \ge 0$$

a standard maximization problem?

Solution To see whether this linear programing problem is a standard linear programming problem, check the requirements above.

The objective function is maximized ← $\boxed{\text{Maximize}}$ $\boxed{z = 5x_1 + 6x_2}$ → The objective function has the form $z = a_1x_1 + a_2x_2$

subject to

$$\boxed{\begin{array}{l} 2x_1 + x_2 \le 4 \\ x_1 + 2x_2 \le 4 \end{array}}$$

→ All constraints have the form $b_1x_1 + b_2x_2 \le c$ where c is nonnegative.

$$\boxed{x_1 \ge 0,\ x_2 \ge 0}$$

→ Decision variables are nonnegative

Since all the requirements are met, this is a standard minimization problem.

1. Is the linear programming problem

$$\text{Maximize } z = 3x_1 + 4x_2$$

subject to

$$x_1 + x_2 \le 40$$

$$x_1 + 2x_2 \le 60$$

$$x_1 \ge 0,\ x_2 \ge 0$$

a standard maximization problem?

Question 2 – What are slack variables?

Key Terms

Slack variables	Initial simplex tableau
Initial simplex tableau	Indicator row

Summary

Slack variables are extra variables that are nonnegative that are added to constraints to change them from inequalities to equalities. For instance, in the standard maximization problem below,

$$\text{Maximize} \quad z = 3x_1 + 4x_2 \text{ subject to}$$

$$2x_1 + 3x_2 \le 6$$

$$2x_1 + x_2 \le 4$$

$$x_1 \ge 0, x_2 \ge 0$$

The constraints are

$$2x_1 + 3x_2 \le 6$$

$$2x_1 + x_2 \le 4$$

We can change these inequalities to equalities by adding a nonnegative number to the left side of each inequality. If the slack variables are called s_1 and s_2, then the inequalities become

$$2x_1 + 3x_2 + s_1 = 6$$

$$2x_1 + x_2 + s_2 = 4$$

The objective function $z = 3x_1 + 4x_2$ is already an equation, but we can write it with all of the variables on the left side as

$$-3x_1 - 4x_2 + z = 0$$

Put all of these into a system of linear equations and we get

$$\begin{aligned} 2x_1 + 3x_2 + s_1 \qquad\quad &= 6 \\ 2x_1 + x_2 \qquad + s_2 \quad &= 4 \\ -3x_1 - 4x_2 \qquad\quad + z &= 0 \end{aligned}$$

This is a system of three linear equations in five variables. The corresponding augmented matrix is

$$
\begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\
\left[\begin{array}{ccccc|c}
2 & 3 & 1 & 0 & 0 & 6 \\
2 & 1 & 0 & 1 & 0 & 4 \\
\hline
-3 & -4 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}
$$

This matrix is called the initial simplex tableau. The bottom row in the tableau always originates from the objective function and is called the indicator row.

Notes

Guided Example

Find the initial simplex tableau for the linear programming problem.

Maximize $z = 2x_1 + 4x_2$

subject to

$$x_1 + x_2 \leq 10$$

$$15x_1 + 10x_2 \leq 120$$

$$x_1 \geq 0,\ x_2 \geq 0$$

Solution Add slack variables to the constraints to give

$$x_1 + x_2 + s_1 = 10$$

$$15x_1 + 10x_2 + s_2 = 120$$

Practice

1. Find the initial simplex tableau for the linear programming problem.

Maximize $z = 3x_1 + 4x_2$

subject to

$$x_1 + x_2 \leq 40$$

$$x_1 + 2x_2 \leq 60$$

$$x_1 \geq 0,\ x_2 \geq 0$$

The objective function can be rewritten as $-2x_1 - 4x_2 + z = 0$. Putting these equations together gives the system $$\begin{aligned} x_1 + x_2 + s_1 &= 10 \\ 15x_1 + 10x_2 + s_2 &= 120 \\ -2x_1 - 4x_2 + z &= 0 \end{aligned}$$ Write this system in matrix form to give the initial simplex tableau $$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc\|c} 1 & 1 & 1 & 0 & 0 & 10 \\ 15 & 10 & 0 & 1 & 0 & 120 \\ \hline -2 & -4 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$	

<u>Notes</u>

Question 3 – How do you find a basic feasible solution?

<u>Key Terms</u>

Basic feasible solution   Basic variable

Nonbasic variable

<u>Summary</u>

A simplex tableau represents a system of equations in many variables. Typically, there are more variables than equations which means that the system is dependent has an infinite number of solutions. For instance, the initial simplex tableau

$$\begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 2 & 3 & 1 & 0 & 0 & 6 \\ 2 & 1 & 0 & 1 & 0 & 4 \\ \hline -3 & -4 & 0 & 0 & 1 & 0 \end{array}\right]$$

has five variables and three equations. If we were to solve this system, we would be able to solve for three of the variables in terms of the other two variables. For this system, it would be easy to solve for s_1, s_2, and z since the columns corresponding to those variables contain ones and zeros. Because of this, these variables are called basic variables. The other two variables, x_1 and x_2, are called nonbasic variables. The numbers in these columns typically do not consist of ones and zeros.

Any simplex tableau corresponds to a solution that may be found by setting the nonbasic variables equal to zero. This has the effect of eliminating those columns from the system. If we set x_1 and x_2 equal to zero, we can cover up those columns and read a solution from the remaining part of the matrix.

$$\begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 2 & 3 & 1 & 0 & 0 & 6 \\ 2 & 1 & 0 & 1 & 0 & 4 \\ \hline -3 & -4 & 0 & 0 & 1 & 0 \end{array}\right]$$

The basic feasible solution corresponding to this simplex tableau is $x_1 = 0$, $x_2 = 0$, $s_1 = 6$, $s_2 = 4$ and $z = 0$. If we had solved this problem geometrically, this point would have corresponded to the corner point at the origin.

Other basic feasible solutions are obtained by putting ones and zeros in different columns and setting the nonbasic variables equal to zero. This is done by carrying out row operations. For instance, suppose we apply the row operations below on the matrix above:

$$\begin{array}{r} \tfrac{1}{3}R_1 \to R_1 \\ -1R_1 + R_2 \to R_2 \\ 4R_1 + R_3 \to R_3 \end{array} \qquad \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} \tfrac{2}{3} & 1 & \tfrac{1}{3} & 0 & 0 & 2 \\ \tfrac{4}{3} & 0 & -\tfrac{1}{3} & 1 & 0 & 2 \\ \hline -\tfrac{1}{3} & 0 & \tfrac{4}{3} & 0 & 0 & 8 \end{array}\right] \end{array}$$

The solution corresponding to this simplex tableau is $x_1 = 0$, $x_2 = 2$, $s_1 = 0$, $s_2 = 2$ and $z = 8$. This point also corresponds to a corner point on the feasible region.

By selecting different basic and nonbasic variables, we can find every corner point on the feasible region for the linear programming problem.

Notes

Guided Example

Find the basic feasible solution corresponding to the simplex tableau

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 10 \\ 5 & 0 & -10 & 1 & 0 & 20 \\ \hline 2 & 0 & 4 & 0 & 1 & 40 \end{array}\right] \end{array}$$

Solution The nonbasic variables are x_1 and s_1. Cover the variables in the simplex tableau:

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 10 \\ 5 & 0 & -10 & 1 & 0 & 20 \\ \hline 2 & 0 & 4 & 0 & 1 & 40 \end{array}\right] \end{array}$$

This gives the basic feasible solution $x_1 = 0$, $x_2 = 10$, $s_1 = 0$, $s_2 = 20$ and $z = 40$.

Practice

1. Find the basic feasible solution corresponding to the simplex tableau

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 2 & 1 & -1 & 0 & 40 \\ 1 & 1 & 0 & \tfrac{1}{2} & 0 & 20 \\ \hline 0 & -1 & 0 & \tfrac{3}{2} & 1 & 120 \end{array}\right] \end{array}$$

Question 4 – How do you get the optimal solution to a standard maximization problem with the Simplex Method?

Key Terms

Pivot row Pivot column

Pivot

Summary

The Simplex Method is a technique for discovering which variables should be basic and which should be nonbasic. It is an iterative procedure which will find the solution of any standard minimization problem.

1. Make sure the linear programming problem is a standard maximization problem.

2. Convert each inequality to an equality by adding a slack variable. Each inequality must have a different slack variable. Each constraint will now be an equality of the form

$$b_1x_1 + b_2x_2 + \cdots + b_nx_n + s = c$$

   where s is the slack variable for the constraint. If more than one slack variable is needed, use subscripts like $s_1, s_2, \ldots$

3. Rewrite the objective function $z = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ by moving all of the variables to the left side. After rewriting the equation, the function will have the form

$$-a_1x_1 - a_2x_2 - \cdots - a_nx_n + z = 0.$$

4. Convert the equations from steps 2 and 3 to an initial simplex tableau. Put the equation from step 3 in the bottom row of the tableau and all other equations above it. The bottom row is called the indicator row.

5. Find the entry in the indicator row that is most negative. If two of the entries are most negative and equal, pick the entry that is farthest to the left. The column with this entry is called the pivot column.

6. For each row except the last row, divide the entry in the last column by the entry in the pivot column. Ignore any rows where the entry in the pivot column is negative. The row with the smallest non-negative quotient is the pivot row. If more than one row has the same smallest quotient, the higher of the rows is the pivot row.

7. The pivot is the entry where the pivot row and pivot column intersect. Multiply the pivot row by the reciprocal of the pivot to change it to a 1.

8. To change the rest of the pivot column to zeros, multiply the pivot row by constants and add them to the other rows in the tableau. Replace those rows with the appropriate sums. When complete, the pivot should be a one, and the rest of the pivot column should be zeros.

9. If the indicator does not contain any negative entries, this tableau corresponds to the optimum solution. In this case, cover the nonbasic variables (set the nonbasic variables equal to zero), and read off the solution for the basic variables. Otherwise, repeat steps 5 through 9 until the indicator row contains no negative numbers.

Notes

Guided Example

Find the optimal solution for the linear programming problem below using the simplex method.

$$\text{Maximize z} = 60x_1 + 50x_2$$

subject to

$$x_1 + x_2 \le 100$$

$$x_1 + 2x_2 \le 180$$

$$x_1 \ge 0,\ x_2 \ge 0$$

Solution Start by adding slack variables to each inequality to change it into an equation:

$$x_1 + x_2 + s_1 = 100$$

$$x_1 + 2x_2 + s_2 = 180$$

Rewrite the objective function to put all of its variable terms on the left side of the equation:

$$-60x_1 - 50x_2 + z = 0$$

Put these equations into the initial simplex tableau with the objective function equation in the bottom row:

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 100 \\ 1 & 2 & 0 & 1 & 0 & 180 \\ \hline -60 & -50 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The pivot column will be the first column since the most negative indicator is in the first column. The pivot row is the second row since the ratio $\frac{100}{1}$ is smaller than $\frac{180}{1}$. This makes the pivot the number 1 in the first row, first column.

Since the pivot is already a 1 (if it is not divide the row by the pivot to make it a 1), use the row

Practice

1. Find the optimal solution for the linear programming problem below using the simplex method.

$$\text{Maximize z} = 3x_1 + 4x_2$$

subject to

$$x_1 + x_2 \le 40$$

$$x_1 + 2x_2 \le 60$$

$$x_1 \ge 0,\ x_2 \ge 0$$

operations $-1R_1+R_2 \to R_2$ and $60R_1+R_3 \to R_3$ to put zeros in the rest of the pivot column: $$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc\|c} 1 & 1 & 1 & 0 & 0 & 100 \\ 0 & 1 & -1 & 1 & 0 & 80 \\ \hline 0 & 10 & 60 & 0 & 1 & 6000 \end{array}\right] \end{array}$$ The indicator row does not contain any negative indicators, so this tableau is the final tableau. It corresponds to the solution $x_1 = 100$, $x_2 = 0$, and $z = 6000$. If the indicator row had still contained a negative indicator, we would have picked a new pivot and used row operations to change the pivot to a one. More row operations would then be used to make the other entries in the pivot column into zeros.	

Question 5 – How do you find the optimal solution for an application?

<u>Key Terms</u>

<u>Summary</u>

A linear programming application can be broken down into two parts. First, you need to set up the application by writing out the variables, objective function, and constraints. Once the linear programming problem is written down and determined to be a standard maximization problem, we can solve the problem with the Simplex Method.

<u>Notes</u>

Guided Example

Carrie Green is working to raise money for the homeless by sending information letters and making follow up calls to local labor organizations and church groups. She discovers that each church group requires 2 hours of letter writing and 1 hour of follow up while for each labor union she needs 2 hours of letter writing and 3 hours of follow up. Carrie can raise \$100 from each church group and \$200 from each union local, and she has a maximum of 16 hours of letter writing time and a maximum of 12 hours of follow up time available per month. Determine the most profitable mixture of groups she should contact and the most money she can raise.

Solution Follow the steps outlined above.

Set Up The Linear Programming Problem - To get started, we need to identify the variables in this problem. Since the problem asks us to "determine the most profitable mixture of groups she should contact", let's define

C: number of church groups to contact

U: number of union locals to contact

With these two variables defined, let's find the objective function. The problem statement asks us to determine "the most money she can raise". Since she can raise \$100 from each church group and \$200 from each union local, the objective function must be

$$Z = 100C + 200U$$

Now look for the factors that constrain her fundraising. Two pieces of information are evident:

she has a maximum of 16 hours of letter writing time

she has a maximum of 12 hours of follow up time

This leads me to write

$$\text{total amount of letter writing time} \le 16 \text{ hours}$$
$$\text{total amount of follow up time} \le 12 \text{ hours}$$

Let's tackle the first piece of information. Since it regards letter writing, let's find the information for letter writing.

each church group requires 2 hours of letter writing

each labor union she needs 2 hours of letter writing

So, if we have C church groups and U union locals, we can write

$$2C + 2U \le 16$$

Following a similar strategy for follow up leads us to

$$C + 3U \le 12$$

Now that we have the objective function and the constraints, we can write out the linear programming problem:

$$\begin{aligned}&\text{Maximize} \quad Z = 100C + 200U\\&\text{subject to}\\&\quad 2C + 2U \le 16\\&\quad C + 3U \le 12\\&\quad C \ge 0, U \ge 0\end{aligned}$$

Now we can carry out the simplex method.

Carry Out The Simplex Method - Rewrite the problem with two slack variables:

$$\begin{aligned}&\text{Maximize} \quad Z = 100C + 200U\\&\text{subject to}\\&\quad 2C + 2U + s_1 \qquad = 16\\&\quad C + 3U \qquad + s_2 = 12\\&\quad C \ge 0, U \ge 0, s_1 \ge 0, s_2 \ge 0\end{aligned}$$

The initial tableau is

$$\begin{array}{c} \begin{array}{ccccc} C & U & s_1 & s_2 & Z \end{array} \\ \left[\begin{array}{ccccc|c} 2 & 2 & 1 & 0 & 0 & 16 \\ 1 & 3 & 0 & 1 & 0 & 12 \\ \hline -100 & -200 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The pivot column is the second column since -200 is the most negative entry in the indicator row. The pivot row is the second row since $\frac{12}{3} < \frac{16}{2}$. Therefore, we need to change the 3 in the pivot entry to a 1 by performing $\frac{1}{3}R_2 \to R_2$:

$$\begin{array}{c} \begin{array}{cccccc} C & U & s_1 & s_2 & Z & \end{array} \\ \left[\begin{array}{ccccc|c} 2 & 2 & 1 & 0 & 0 & 16 \\ \frac{1}{3} & 1 & 0 & \frac{1}{3} & 0 & 4 \\ \hline -100 & -200 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Now we need to put 0's in the rest of the pivot column by performing $-2R_2 + R_1 \to R_1$ and $200R_2 + R_3 \to R_3$:

$$\begin{array}{c} \begin{array}{cccccc} C & U & s_1 & s_2 & Z & \end{array} \\ \left[\begin{array}{ccccc|c} \frac{4}{3} & 0 & 1 & -\frac{2}{3} & 0 & 8 \\ \frac{1}{3} & 1 & 0 & \frac{1}{3} & 0 & 4 \\ \hline -\frac{100}{3} & 0 & 0 & \frac{200}{3} & 1 & 800 \end{array}\right] \end{array}$$

Since there is a negative number in the indicator row, we need to pivot again. The new pivot is the $\frac{4}{3}$ in the first row, first column. We begin by changing the pivot to a 1 by performing $\frac{3}{4}R_1 \to R_1$:

$$\begin{array}{c} \begin{array}{cccccc} C & U & s_1 & s_2 & Z & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & \frac{3}{4} & -\frac{2}{4} & 0 & 6 \\ \frac{1}{3} & 1 & 0 & \frac{1}{3} & 0 & 4 \\ \hline -\frac{100}{3} & 0 & 0 & \frac{200}{3} & 1 & 800 \end{array}\right] \end{array}$$

To put 0's in the rest of the column, perform $-\frac{1}{3}R_1 + R_2 \to R_2$ and $\frac{100}{3}R_1 + R_3 \to R_3$:

$$\begin{array}{c} \begin{array}{cccccc} C & U & s_1 & s_2 & Z & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & \frac{3}{4} & -\frac{2}{4} & 0 & 6 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & 0 & 2 \\ \hline 0 & 0 & 25 & 50 & 1 & 1000 \end{array}\right] \end{array}$$

Since there are no negative numbers in the indicator row, we have arrived at the maximum amount of money raised, \$1000. This is done by contacting 6 church groups and 2 union locals.

Practice

A convenience store sells three types of juices: grape, cranberry, and mango. It earns \$0.60, \$0.76, and \$0.99 in profit on each bottle of the three juices, respectively. It can stock no more than 400 bottles in the store each week. Typically, at least twice as many cranberry bottles are sold as mango bottles. The company never sells more than 100 bottles of grape juice in a week. How many bottles of each juice should the store stock to maximize profit?

Section 4.4 The Simplex Method and the Standard Minimization Problem

Question 1 – What is a standard minimization problem?

Question 2 – How is the standard minimization problem related to the dual standard maximization problem?

Question 3 - How do you apply the Simplex Method to a standard minimization problem?

Question 4 - How do you apply the Simplex Method to a minimization application?

Question 1 – What is a standard minimization problem?

Key Terms

Standard maximization problem

Summary

A standard minimization problem is a type of linear programming problem in which the objective function is to be minimized and has the form

$$w = d_1 y_1 + d_2 y_2 + \cdots + d_n y_n$$

where $d_1, \ldots, d_n$ are real numbers and $y_1, \ldots, y_n$ are decision variables. The decision variables must represent non-negative values. The other constraints for the standard minimization problem have the form

$$e_1 y_1 + e_2 y_2 + \cdots + e_n y_n \geq f$$

where $e_1, \ldots, e_n$ and f are real numbers and $f \geq 0$.

The standard minimization problem is written with the decision variables $y_1, \ldots, y_n$, but any letters could be used as long as the standard minimization problem and the corresponding dual maximization problem do not share the same variable names.

Notes

Guided Example	Practice
Rewrite the linear programming problem so that it is a standard minimization problem. Minimize $w = 16y_1 + 14y_2 + 12y_3$ subject to $y_1 + y_2 + y_3 \geq 6000$ $y_3 \leq \frac{2}{3} y_2$ $y_1 \geq 0.25(y_1 + y_2 + y_3)$ $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$ **Solution** The objective function has the proper form. The inequality fits the form needed for a standard minimization. In this format the variables appear linearly on the left side of the inequality and are greater than or equal to a nonnegative number. Each of the inequalities must have this form to apply the Simplex Method. The second constraint has variables on both sides of the inequality. To put it into the proper form, subtract y_3 from both sides to give $0 \leq \frac{2}{3} y_2 - y_3$ Flip flopping the sides and inequality give an equivalent inequality that is in the proper format: $\frac{2}{3} y_2 - y_3 \geq 0$ To put the third inequality into the proper format, subtract $0.25(y_1 + y_2 + y_3)$ from both sides to yield $y_1 - 0.25(y_1 + y_2 + y_3) \geq 0$ Remove the parentheses and combine like terms to get , Now add these modified constraints to the linear programming problem,	1. Rewrite the linear programming problem so that it is a standard minimization problem. Minimize $w = .06y_1 + .04y_2 + .02y_3$ subject to $y_1 + y_2 + y_3 \geq 1000$ $y_3 \leq \frac{1}{2} y_2$ $y_2 \geq 0.5(y_1 + y_3)$ $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$

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Minimize $w = 16y_1 + 14y_2 + 12y_3$
subject to
$$y_1 + y_2 + y_3 \geq 6000$$
$$\tfrac{2}{3}y_2 - y_3 \geq 0$$
$$0.75y_1 - 0.25y_2 - 0.25y_3 \geq 0$$
$$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$$

Notice that all of the variables are on the left side of the greater than sign in the inequalities. Additionally, the right side of each inequality is nonnegative.
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Question 2 – How is the standard minimization problem related to the dual standard maximization problem?

Key Terms

Dual problem   Transpose

Summary

The linear programming problem

Minimize $z = \frac{7}{4}y_1 + y_2$

subject to

$$y_1 + 4y_2 \geq 80$$

$$7y_1 + 4y_2 \geq 280$$

$$y_1 \geq 0, \; y_2 \geq 0$$

is a standard minimization problem. The related dual maximization problem is found by forming a matrix before the objective function is modified or slack variables are added to the constraints. The entries in this matrix are formed from the coefficients and constants in the constraints and objective function:

Coefficients from the first constraint

Coefficients from the second constraint

Coefficients from the objective function

$$\left[\begin{array}{cc|c} \boxed{1 \quad 4} & & \boxed{80} \\ \boxed{7 \quad 4} & & \boxed{280} \\ \hline \boxed{\tfrac{7}{4} \quad 1} & & \boxed{0} \end{array}\right]$$

Constant from the first constraint

Constant from the second constraint

No constants in the objective function

To find the coefficients and constants in the dual problem, switch the rows and columns. In other words, make the rows in the matrix above become the columns in a new matrix,

$$\left[\begin{array}{cc|c} 1 & 7 & \frac{7}{4} \\ 4 & 4 & 1 \\ \hline 80 & 280 & 0 \end{array}\right]$$

This new matrix is called the transpose of the original matrix. The values in the new matrix help us to form the constraints and objective function in a standard maximization problem:

$$\left[\begin{array}{cc|c} 1 & 7 & \frac{7}{4} \\ 4 & 4 & 1 \\ \hline 80 & 280 & 0 \end{array}\right]$$

$$x_1 + 7x_2 \le \tfrac{7}{4}$$

$$4x_1 + 4x_2 \le 1$$

$$\text{Maximize } z = 80x_1 + 280x_2$$

Notice the inequalities have switched directions since the dual problem is a standard maximization problem and the names of the variables are different from the original minimization problem. Putting these details together with non-negativity constraints, we get the standard maximization problem

$$\text{Maximize } z = 80x_1 + 280x_2$$

subject to

$$x_1 + 7x_2 \le \tfrac{7}{4}$$

$$4x_1 + 4x_2 \le 1$$

$$x_1 \ge 0, x_2 \ge 0$$

This strategy works in general to find the dual problem.

Notes

Guided Example

Find the dual maximization problem to the standard minimization problem below.

Minimize $w = 4y_1 + 6y_2 + 8y_3$

subject to

$$5y_1 + 10y_2 + 12y_3 \ge 100$$
$$2y_1 + 3y_2 + 5y_3 \ge 300$$
$$-0.5y_1 + 0.75y_2 + 0.75y_3 \ge 0$$
$$y_1 \ge 0,\ y_2 \ge 0, y_3 \ge 0$$

Solution Form a matrix where the first three rows correspond to the three constraints and the fourth row corresponds to the objective function:

$$\left[\begin{array}{ccc|c} 5 & 10 & 12 & 100 \\ 2 & 3 & 5 & 300 \\ -0.5 & 0.75 & 0.75 & 0 \\ \hline 4 & 6 & 8 & 0 \end{array}\right]$$

The transpose of this matrix is

Practice

1. Find the dual maximization problem to the standard minimization problem below.

Minimize $w = 2y_1 + y_2$

subject to

$$y_1 + y_2 \ge 1$$
$$2y_1 + 4y_2 \ge 3$$
$$y_1 \ge 0,\ y_2 \ge 0$$

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$$\left[\begin{array}{ccc|c} 5 & 2 & -0.5 & 4 \\ 10 & 3 & 0.75 & 6 \\ 12 & 5 & 0.75 & 8 \\ \hline 100 & 300 & 0 & 0 \end{array}\right]$$

Use this matrix to write out the dual standard maximization problem with the variables x_1, x_2, x_3, and *z:*

Maximize $\text{z} = 100x_1 + 300x_2$

subject to

$$5x_1 + 2x_2 - 0.5x_3 \le 4$$

$$10x_1 + 3x_2 + 0.75x_3 \ge 6$$

$$12x_1 + 5x_2 + 0.75x_3 \ge 8$$

$$x_1 \ge 0,\ x_2 \ge 0, x_3 \ge 0$$

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<u>Notes</u>

Question 3 – How do you apply the Simplex Method to a standard minimization problem?

Key Terms

Summary

To solve a standard minimization problem with the dual maximum problem

1. Make sure the minimization problem is in standard form. If it is not in standard form, modify the problem to put it in standard form.

2. Find the dual standard maximization problem.

3. Apply the Simplex Method to solve the dual maximization problem.

4. Once the final simplex tableau has been calculated, the minimum value of the standard minimization problem's objective function is the same as the maximum value of the standard maximization problem's objective function.

5. The solution to the standard minimization problem is found in the bottom row of the final simplex tableau in the columns corresponding to the slack variables.

Notes

Guided Example

Use the Simplex Method to solve

$$\text{Minimize } w = 5y_1 + 2y_2$$

subject to

$$2y_1 + 3y_2 \geq 6$$

$$2y_1 + y_2 \geq 7$$

$$y_1 \geq 0,\ y_2 \geq 0$$

Solution Start by finding the dual maximization problem. The matrix for the minimization problem is

$$\left[\begin{array}{cc|c} 2 & 3 & 6 \\ 2 & 1 & 7 \\ \hline 5 & 2 & 0 \end{array}\right]$$

The transpose of this matrix is

$$\left[\begin{array}{cc|c} 2 & 2 & 5 \\ 3 & 1 & 2 \\ \hline 6 & 7 & 0 \end{array}\right]$$

This gives a dual maximization problem

$$\text{Maximize } z = 6x_1 + 7x_2$$

subject to

$$2x_1 + 2x_2 \leq 5$$

$$3x_1 + x_2 \leq 2$$

$$x_1 \geq 0,\ x_2 \geq 0$$

The initial simplex tableau is

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 2 & 2 & 1 & 0 & 0 & 5 \\ 3 & 1 & 0 & 1 & 0 & 2 \\ \hline -6 & -7 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The pivot is the entry in the second row, second column. Since it is already a 1, use the row operations $-2R_2 + R_1 \rightarrow R_1$ and $7R_2 + R_3 \rightarrow R_3$ to put zeros above and below the pivot:

$$
\begin{array}{c}
\begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\
\left[\begin{array}{ccccc|c}
-4 & 0 & 1 & -2 & 0 & 1 \\
3 & 1 & 0 & 1 & 0 & 2 \\
\hline
15 & 0 & 0 & 7 & 1 & 14
\end{array}\right]
\end{array}
$$

Since there are no negative numbers in the indicator row, we can read the solution from this tableau. If any of the indicators in the bottom row had been negative, we would need to pick another pivot and carry out the steps to get ones and zeros in the pivot column. The solution from the final tableau for the maximization problem is $x_1 = 0$, $x_2 = 2$ and $z = 14$. The minimization problem's solution is found under the slack variable yielding $y_1 = 0$, $y_2 = 7$, and $w = 14$.

Practice

1. Use the Simplex Method to solve

$$\text{Minimize w} = 8y_1 + 16y_2$$

subject to

$$y_1 + 5y_2 \geq 9$$

$$2y_1 + 2y_2 \geq 10$$

$$y_1 \geq 0,\ y_2 \geq 0$$

Question 4 – How do you apply the Simplex Method to a minimization application?

Key Terms

Summary

As with maximization applications, a minimization application can be broken down into two parts. First, you need to set up the application by writing out the variables, objective function, and constraints. Once the linear programming problem is written down and determined to be a standard minimization problem, we can solve the problem with the Simplex Method. For the minimization problem this will require you to find the dual maximization problem and then solve the maximization problem with the techniques from the previous questions. Remember to find the solution to the minimization problem in the indicator row and in the columns corresponding to the slack variables.

Notes

Guided Example

The chemistry department at a local college decides to stock at least 800 small test tubes and 500 large test tubes. It wants to buy at least 2100 test tubes to take advantage of a special price. Since the small tubes are broken twice as often as the large, the department will order at least twice as many small tubes as large. If the small test tubes cost 15¢ each and large ones, made of a cheaper glass, cost 12¢ each, how many of each size should be ordered to minimize cost?

Solution Start by examining the question, "How many of each size should be ordered to minimize cost?"

This tells us two things:

1. We'll be minimizing cost.
2. The variables are the number of test tubes of each type.

Start by defining exactly what the variables will represent:

y_1 : number of large test tubes to order
y_2 : number of small test tubes to order

In minimization problems we generally use y's as variables (x's for a maximization problem). Next, we need to relate the variables to the cost that we are minimizing. Locating the information about cost (small test tubes cost 15¢ each and large ones, made of a cheaper glass, cost 12¢ each), we can write

$$\text{Minimize } C = 12y_1 + 15y_2$$

where C is in cents.

With the objective function done, we can now move onto the constraints.

"Decides to stock at least 800 small test tubes and 500 large test tubes" means

$$y_2 \geq 800$$
$$y_1 \geq 500$$

"Buy at least 2100 test tubes to take advantage of a special price" means

$$y_1 + y_2 \geq 2100$$

Finally, the statement "the department will order at least twice as many small tubes as large" means

$$y_1 \geq 2y_2$$

This makes sense since it indicates that the small tubes are more than double the large tubes.

$$\begin{aligned} &\text{Minimize } C = 12y_1 + 15y_2 \\ &\text{subject to} \\ &\qquad y_2 \geq 800 \\ &\qquad y_1 \geq 500 \\ &\qquad y_1 + y_2 \geq 2100 \\ &\qquad y_1 \geq 2y_2 \end{aligned}$$

Before we can start using the simplex method, we need to rewrite the last constraint in the proper form, $y_1 - 2y_2 \geq 0$.

Now that have the linear programming problem in the standard linear form, we need to convert to the dual:

$$\begin{bmatrix} 0 & 1 & 800 \\ 1 & 0 & 500 \\ 1 & 1 & 2100 \\ 1 & -2 & 0 \\ 12 & 15 & 0 \end{bmatrix} \xrightarrow[\textit{TRANSPOSE}]{} \begin{bmatrix} 0 & 1 & 1 & 1 & 12 \\ 1 & 0 & 1 & -2 & 15 \\ 800 & 500 & 2100 & 0 & 0 \end{bmatrix}$$

Converting this to a standard maximum problem yields

$$\begin{aligned} &\text{Maximize } z = 800x_1 + 500x_2 + 2100x_3 \\ &\text{subject to} \\ &\qquad x_2 + x_3 + x_4 \geq 12 \\ &\qquad x_1 + x_3 - 2x_4 \geq 15 \\ &\qquad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0 \end{aligned}$$

Adding slack variables to the constraints and rewriting the objective function we get

$$\begin{aligned} x_2 + x_3 + x_4 + s_1 &= 12 \\ x_1 + x_3 - 2x_4 + s_2 &= 15 \\ -800x_1 - 500x_2 - 2100x_3 + z &= 0 \end{aligned}$$

The initial tableau is

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccccc|c} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 12 \\ 1 & 0 & 1 & -2 & 0 & 1 & 0 & 15 \\ \hline -800 & -500 & -2100 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The pivot column is the third column since -2100 is the most negative indicator. The pivot row is the first row since the ratio 12/1 is smallest. The pivot entry is already 1 so we need to make the rest of the column 0. To do this, we carry out $-1R_1 + R_2 \to R_2$ and $2100R_1 + R_3 \to R_3$. The new matrix is

$$\left[\begin{array}{ccccccc|c} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 12 \\ 1 & -1 & 0 & -3 & -1 & 1 & 0 & 3 \\ \hline -800 & 1600 & 0 & 2100 & 2100 & 0 & 1 & 25200 \end{array}\right]$$

There is a negative in the indicator row, so pick it as the pivot column. Only the ratio 3/1 makes sense so the element in the second row, first column is the pivot. To make a zero below it, $800R_2 + R_3 \to R_3$. The resulting matrix is

$$\left[\begin{array}{ccccccc|c} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 12 \\ 1 & -1 & 0 & -3 & -1 & 1 & 0 & 3 \\ \hline 0 & 800 & 0 & -300 & 1300 & 800 & 1 & 27600 \end{array}\right]$$

Now the fourth column and first row is the pivot (you can't choose a negative in the ratio denominator). To put zero in the rest of the column, carry out $3R_1 + R_2 \to R_2$ and $300R_1 + R_3 \to R_3$. The resulting matrix is

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccccc|c} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 12 \\ 1 & 2 & 3 & 0 & 2 & 1 & 0 & 39 \\ \hline 0 & 1100 & 300 & 0 & 1600 & 800 & 1 & 31200 \end{array}\right] \end{array}$$

The solution to the dual problem (the minimization problem) is found under the slack variables so $y_1 = 1600$, $y_2 = 800$ and $z = 31200$.

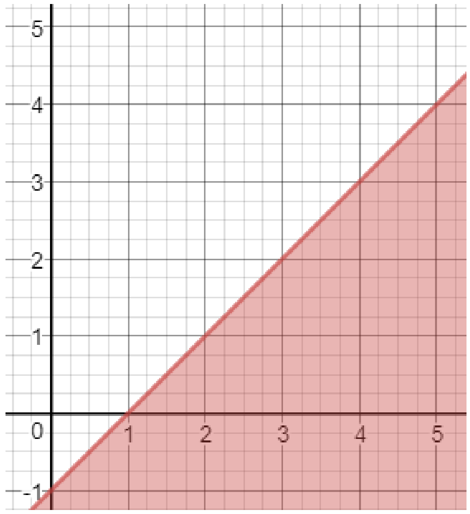
Practice

1. An animal food must provide at least 54 units of vitamins and 60 calories per serving. One gram of soybean meal provides 2.5 units of vitamins and 5 calories. One gram of meat byproducts provides 4.5 units of vitamins and 3 calories. One gram of grain provides 5 units of vitamins and 10 calories. A gram of soybean meal costs 8¢, a gram of meat byproducts 12¢, and a gram of grain 10¢. What mixture of these three ingredients will provide the required vitamins and calories at minimum cost?

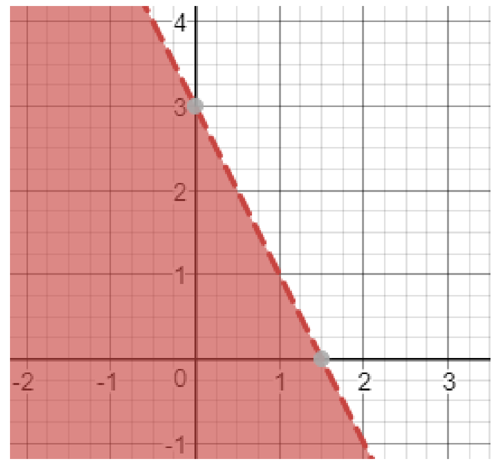
Chapter 4 Solutions

Section 4.1

Question 1 1)

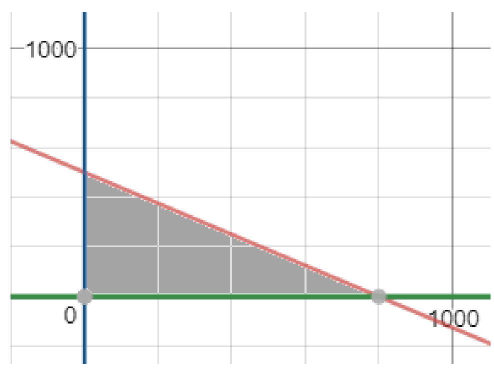


2)

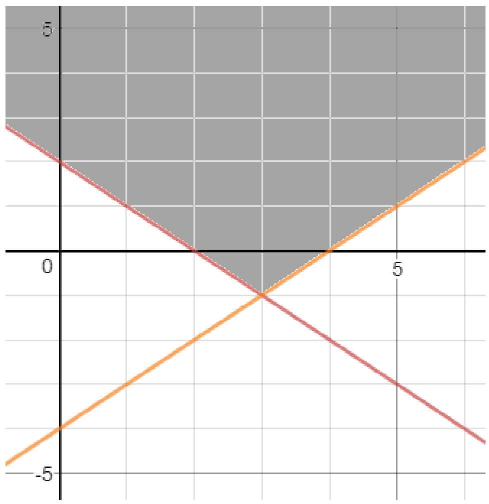


3)

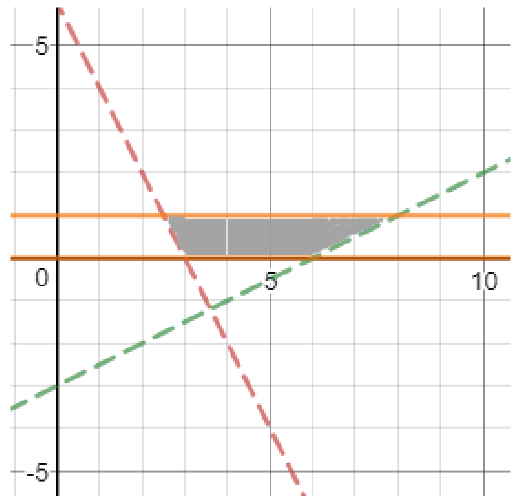
4)



Question 2 1)

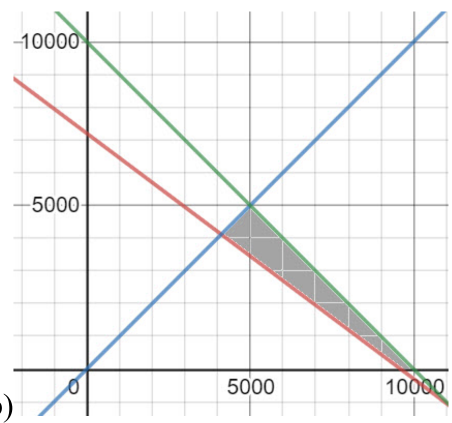


2)



3a) $x + y \le 10{,}000,\ .015x + .02y \ge 144,\ x \ge y,\ x \ge 0,\ y \ge 0$

3b)



Section 4.2

Question 1 1a) $z = 3x_1 + 4x_2$, 1b) $x_1 + x_2 \le 40,\ x_1 + 2x_2 \le 60$, 1c) $x_1 \ge 0,\ x_2 \ge 0$

Question 2 1) Maximum of 140 at (20, 20) 2) All points on the line connecting $\left(\frac{1}{2},\frac{1}{2}\right)$ and $\left(\frac{3}{2},0\right)$ yield the same minimum value of $z = 1.5$.

Section 4.3

Question 1 1) Yes

Question 2 1)
$$\begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 140 \\ 1 & 2 & 0 & 1 & 0 & 60 \\ \hline -3 & -4 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Question 3 1) $x_1 = 20, x_2 = 0, s_1 = 40, s_2 = 0, z = 120$

Question 4 1) $x_1 = 20, x_2 = 20, z = 140$

Question 5 1) The linear programming problem Maximize $P = 0.6G + 0.76C + 0.99M$ subject to $G \le 100$, $G + C + M \le 400$, $2M \le C$ with $G \ge 0, C \ge 0, M \ge 0$ has a solution $G = 0, C = 266\frac{2}{3}, M = 133\frac{1}{3}$.

Section 4.4

Question 1 1) Minimize $w = .06y_1 + .04y_2 + .02y_3$ subject to $y_1 + y_2 + y_3 \ge 1000$, $\frac{1}{2}y_2 - y_3 \ge 0$, $-0.5y_1 + y_2 - 0.5y_3 \ge 0$ with $y_1 \ge 0, y_2 \ge 0, y_3 \ge 0$

Question 2 1) Maximize $z = x_1 + 3x_2$ subject to $x_1 + 2x_2 \le 2$, $x_1 + 4x_2 \le 1$ with $x_1 \ge 0, x_2 \ge 0$

Question 3 1) $y_1 = 4, y_2 = 1, w = 48$

Question 4 1) The linear programming problem Minimize $w = 8y_1 + 12y_2 + 10y_3$ subject to $2.5y_1 + 4.5y_2 + 5y_3 \ge 54$, $5y_1 + 3y_2 + 10y_3 \ge 60$ with $y_1 \ge 0, y_2 \ge 0, y_3 \ge 0$ has solution $y_1 = 0, y_2 = 0, y_3 = 10.8, w = 108$.