## Section 5.1 Simple and Compound Interest

Question 1 - What is simple interest?
Question 2 - What is compound interest?
Question 3 - What is an effective interest rate?
Question 4 - What is continuous compound interest?

Question 1 - What is simple interest?

## Key Terms

Future value Present value
Interest rate Simple interest

## Summary

Simple interest is interest computed on the original principal only. If the present value $P V$ (or principal), in dollars, earns interest at a rate of $r$ for $t$ years, then the interest is

$$
I=P V r t
$$

The future value (also called the accumulated amount or maturity value) is the sum of the principal and the interest. This is the amount the present value grows to after the present value and interest are added.

The future value $F V$ at a simple interest rate $r$ per year is

$$
\begin{aligned}
F V & =P V+P V r t \\
& =P V(1+r t)
\end{aligned}
$$

where $P V$ is the present value that is deposited for $t$ years.
The interest rate $r$ is the decimal form of the interest rate written as a percentage. This means an interest rate of $4 \%$ per year is equivalent to $r=0.04$. Take special care to make sure the time units on the interest rate and time are consistent. If the interest rate is an annual rate, make sure the time is in years. If the interest rate is a monthly rate, make sure the time is in months.

## Notes

## Guided Example

Practice

Find the simple interest on a principal (present value) of \$2000 at an annual interest rate of 3\% for 8 months.

Solution Simple interest is calculated with $I=P V r t$. For this problem, the present value is $P V=2000$ and the interest rate is $r=0.03$. Since the interest rate is an annual rate, the time must be in years so $t=\frac{8}{12}$. Put these values into the formula to give

$$
I=2000(0.03)\left(\frac{8}{12}\right)=40
$$

The simple interest is $\$ 40$.

1. Find the simple interest on a principal (present value) of $\$ 1200$ at an annual interest rate of $8 \%$ for 10 months.

## Guided Example

A loan of $\$ 15,500$ was repaid at the end of 18 months. If an $6 \%$ annual rate of interest was charged, what size repayment check (present value and simple interest) was written?

Solution To find the future value of $\$ 15,500$, use

$$
F V=P V(1+r t)
$$

with a present value $P V=15,500$, an interest rate $r=0.06$ and time $t=\frac{18}{12}$. When these values are substituted, you get

$$
F V=15,500\left(1+(0.06)\left(\frac{18}{12}\right)\right)=16,895
$$

The repayment check would be written for \$16,895.

## Practice

2. A loan of $\$ 12,700$ was repaid at the end of 60 months. If an $9 \%$ annual rate of interest was charged, what size repayment check (present value and simple interest) was written?

If $\$ 1375$ earned simple interest of $\$ 502.56$ in 86 months, what was the simple interest rate?

Solution Substitute the present value $P V=1375$, interest $I=502.56$, and time $t=\frac{86}{12}$ into $I=P V r t$ and solve for $r$ :

$$
\begin{aligned}
502.56 & =1375 \cdot r \cdot\left(\frac{86}{12}\right) \\
\frac{502.56}{1375 \cdot\left(\frac{86}{12}\right)} & =r \\
0.051 & \approx r
\end{aligned}
$$

The interest rate is approximately $5.1 \%$.
3. If $\$ 2000$ earned simple interest of $\$ 345.56$ in 90 months, what was the simple interest rate?

Question 2 - What is compound interest?

## Key Terms

Compound interest Nominal rate
Interest rate per period Conversion period

## Summary

The future value $F V$ of the present value $P V$ compounded over $n$ conversion periods at an interest rate of $i$ per period is

$$
F V=P V(1+i)^{n}
$$

where the interest rate per period is

$$
i=\frac{r}{m}=\frac{\text { nominal rate }}{\text { number of conversion periods in a year }},
$$

and

$$
n=m t=(\text { number of conversion periods in a year })(\text { term in years }) .
$$

Notes

## Guided Example

Suppose that $\$ 25,000$ is invested at $8 \%$ interest. Find the amount of money in the account after 4 years if the interest is compounded quarterly.

Solution Use the compound interest formula,

$$
F V=P V(1+i)^{n}
$$

where the present value is $P V=25,000$, the interest rate per conversion period is $i=\frac{0.08}{4}$ or 0.02 , and the number of periods is $n=4.4$ or 16 . Using these values, you get

$$
F V=25000(1+0.02)^{16} \approx 34,319.64
$$

The account will have approximately $\$ 34,319.64$.

Practice

1. Suppose that $\$ 30,000$ is invested at $7 \%$ interest. Find the amount of money in the account after 18 years if the interest is compounded semiannually.

## Guided Example

Suppose $\$ 5000$ grows to $\$ 8300$ in 7 years. What is the annual interest rate if interest is compounded semiannually?

Solution Use the compound interest formula,

$$
F V=P V(1+i)^{n}
$$

where the present value is $P V=5,000$, the future value is $F V=8300$, and the number of periods is $n=7 \cdot 2$ or 14 . Put these values into the compound interest formula and solve for $i$ :

$$
\begin{aligned}
8300 & =5000(1+i)^{14} \\
\frac{8300}{5000} & =(1+i)^{14} \\
\sqrt[14]{\frac{8300}{5000}} & =\sqrt[14]{(1+i)^{14}} \\
\sqrt[14]{\frac{8300}{5000}} & =1+i \\
\sqrt[14]{\frac{8300}{5000}}-1 & =i \\
0.03686 & \approx i
\end{aligned}
$$

If the rate per conversion period is 0.0369 and there are two conversion periods per year (semiannual), then the nominal rate is $r \approx 0.03686 \cdot 2$ or approximately $7.37 \%$.

## Practice

2. Suppose $\$ 10,000$ grows to $\$ 15,575$ in 5 years. What is the annual interest rate if interest is compounded quarterly?

## Guided Example

## Practice

Find the present value if the future value is $\$ 14,520.35$ and compounded annually at a nominal rate of $1.256 \%$ for 6 years.

Solution Use the compound interest formula,

$$
F V=P V(1+i)^{n}
$$

where the future value is $F V=14,520.35$, the interest rate per period is $i=0.01256$ and the number of periods is $n=6$. Put these values into the compound interest formula and solve for $P V$ :

$$
\begin{aligned}
14520.35 & =P V(1+0.01256)^{6} \\
\frac{14520.35}{(1+0.01256)^{6}} & =P V \\
13472.63 & \approx P V
\end{aligned}
$$

To accumulate $\$ 14520.35$, you would need to start with approximately $\$ 13472.63$.
3. Find the present value if the future value is $\$ 26,500$ and compounded quarterly at a nominal rate of $3.75 \%$ for 20 years.

Question 3 - What is an effective interest rate?

## Key Terms

Effective interest rate
Summary
The effective interest rate is the simple interest rate that leads to the same future value in one year as the nominal interest rate compounded $m$ times per year.

The effective interest rate is

$$
r_{e}=\left(1+\frac{r}{m}\right)^{m}-1
$$

where $r$ is the nominal interest rate, and $m$ is the number of conversion periods per year. Another name for the effective interest rate is the effective annual rate.

The future value $F V$ compounded at an effective interest rate (APY) of $r_{e}$ is

$$
F V=P V\left(1+r_{e}\right)^{t}
$$

where $P V$ is the present value or principal, and $t$ is the term in years.

Notes

## Guided Example

## Practice

Determine the effective rate for $\$ 1000$ invested for 1 year at $7.40 \%$ compounded quarterly.

Solution The effective interest rate $r_{e}$ is

$$
r_{e}=\left(1+\frac{r}{m}\right)^{m}-1
$$

where the nominal rate is $r=0.074$ and the number of compounding periods in a year is $m=4$. Using these values, you get

$$
r_{e}=\left(1+\frac{0.074}{4}\right)^{4}-1 \approx 0.0761
$$

or $7.61 \%$.

1. Determine the effective rate for $\$ 500$ invested for 1 year at $2.2 \%$ compounded monthly.

Question 4 - What is continuous compound interest?

## Key Terms

Continuous interest rate

## Summary

The future value $F V$ of the present value $P V$ compounded continuously at a nominal interest rate of $r$ per year is

$$
F V=P V e^{r t}
$$

where $t$ is the time in years.
The effective rate $r_{e}$ for a continuous rate of $r$ is

$$
r_{e}=e^{r}-1
$$

Notes

## Guided Example

## Practice

Find the future value of $\$ 3000$ compounded continuously at an annual interest rate of $5.1 \%$ for 18 months.

Solution To find the future value for continuous interest, use the formula

$$
F V=P V e^{r t}
$$

with the present value $P V=3000$, interest rate $r=0.051$, and time $t=\frac{18}{12}$ or 1.5 . Using these values, you get

$$
F V=3000 e^{(0.051)(1.5)} \approx 3238.51
$$

The future value of $\$ 3000$ is $\$ 3238.51$.

1. Find the future value of $\$ 5500$ compounded continuously at an annual interest rate of $1.1 \%$ for 36 months.

## Guided Example

How much should be deposited today at a continuous interest rate of $3.5 \%$ to accumulate $\$ 10,500$ in 5 years?

Solution To find the present value for continuous interest, use the formula

$$
F V=P V e^{r t}
$$

with the future value $F V=10,500$, interest rate $r=0.035$, and time $t=5$. Using these values, solve for the present value $P V$ :

$$
\begin{aligned}
& 10500=P V e^{(0.035)(5)} \\
& \frac{10500}{e^{(0.035)(5)}}=P V \\
& 8814.30 \approx P V
\end{aligned}
$$

If you start with approximately $\$ 8814.30$, it will grow to $\$ 10,500$ in 5 years at a continuous interest rate of $3.5 \%$ per year.

## Practice

2. How much should be deposited today at a continuous interest rate of $5.75 \%$ to accumulate $\$ 100,000$ in 40 years?

## Section 5.2 Exponential and Logarithm Functions in Finance

Question 1 - How do you convert between the exponential and logarithmic forms of an equation?

Question 2 - How do you evaluate a logarithm?
Question 3 - How do you solve problems using logarithms?

Question 1 - How do you convert between the exponential and logarithmic forms of an equation?

## Key Terms

Exponential form Logarithmic form

## Summary

Exponential form and logarithmic form are different ways at looking inputs and outputs. The exponential function

$$
y=10^{x}
$$

takes the variable $x$ as its input and outputs the variable $y$. For an input of $x=2$ we get an output of $y=100$ since

$$
100=10^{2}
$$

On a logarithm of base 10 (called a common logarithm), these roles are reversed. The common logarithm must take in $y=100$ and output $x=2$,

$$
2=\log _{10}(100)
$$

For common logarithms, those with base 10, the base on the logarithm is often left out and written as

$$
2=\log (100)
$$

This means that whenever you see a logarithm without a base, it is assumed to have a base of 10 . Let's compare these forms side by side.

## Exponential Form



Logarithmic Form


The same type of relationships exists for exponentials with a positive base and the corresponding logarithm with that base. For all bases $b>0$,

$$
y=b^{x} \text { means that } x=\log _{b}(y)
$$

Notes

## Guided Example

Rewrite each exponential form in logarithmic form.
a. $6^{2}=36$

Solution Start by recognizing the base on the exponential form, 6 . This means the corresponding logarithmic form will be a logarithm base 6. Since the exponential from takes 2 as the input and outputs 36 , the logarithmic form must take in 36 and output 2. This give the logarithmic form,

$$
\log _{6}(36)=2
$$

b. $\quad 3^{-4}=\frac{1}{81}$

Solution The base in the logarithmic form must be 3. Since the exponential form takes input -4 and outputs $\frac{1}{81}$, the logarithmic form must do the opposite,

$$
\log _{3}\left(\frac{1}{81}\right)=-4
$$

c. $e^{z}=Y$

Solution The base on the exponential form is e so the corresponding logarithm is $\log _{e}$ or $\ln$. The exponential form takes in z and out puts Y so the logarithmic form is

$$
\ln (Y)=z
$$

## Practice

1. Rewrite each exponential form in logarithmic form.
a. $\quad 5^{3}=125$
b. $\quad 2^{-4}=\frac{1}{16}$
c. $\quad e^{r t}=\frac{F V}{P V}$

## Guided Example

Rewrite each logarithmic form in exponential form.
a. $\quad \log _{5}(125)=3$

Solution The logarithm will convert to an exponential form with base 5 . Since the logarithm takes in 125 and outputs 3, the exponential form is

$$
5^{3}=125
$$

b. $\quad \log (0.01)=-2$

Solution Since this is a common logarithm, the base is hidden (but equal to 10). Switching the input and outputs in exponential form leads to

$$
10^{-2}=0.01
$$

c. $\quad \log \left(\frac{I}{I_{0}}\right)=R$

Solution In this common logarithm, the group of symbols $\frac{I}{I_{0}}$ form the input and $R$ is the output. The corresponding exponential form is

$$
10^{R}=\frac{I}{I_{0}}
$$

## Practice

2. Rewrite each logarithmic form in exponential form.
a. $\quad \log _{7}(49)=2$
b. $\log (0.001)=-3$
c. $\quad \log (1+Z)=Q$

Question 2 - How do you evaluate a logarithm?

## Key Terms

## Summary

Many logarithms may be calculated by converting them to exponential form. Suppose we want to calculate the value of $\log _{2}(16)$. Start by writing this expression as a logarithmic form,

$$
\log _{2}(16)=?
$$

We could write the output with a variable, but a question mark suffices to indicate what we want to find. If we convert this form to an exponential form with a base of 2,

$$
2^{?}=16
$$

The left-hand side may be written with the base 2 as $2^{4}$. Substitute this expression in place of 16,

$$
2^{?}=2^{4}
$$

Since the exponent on the left side must be 4, this is also the value in the original exponential form,

$$
\log _{2}(16)=4
$$

This strategy works well if we can write the number on the right with the same base as the exponential on the other side of the equation.

If you are not able to write the number on the right with the same base, we can evaluate common logarithms with the LOG key or a natural logarithm with the LN key on a calculator. If the base on the logarithm is not 10 (common logarithm) or e (natural logarithm), we use the Change of Base Formula to find the logarithm.

## Change of Base Formula for Logarithms

For any positive base $a$ and $b$ not equal to 1 ,

$$
\log _{a}(x)=\frac{\log _{b}(x)}{\log _{b}(a)}
$$

where $x$ is a positive number. Typically, the logarithms on the right-side are done as natural or common logarithms so they can be evaluated on any calculator.

## Guided Example

Evaluate each logarithm without a calculator.
a. $\quad \log _{4}(64)$

Solution Write the logarithm in logarithmic form with? forming the output of the logarithm,

$$
\log _{4}(64)=?
$$

Convert this to exponential form,

$$
4^{?}=64
$$

Since $4^{3}=64$, the value of ? is 3 and

$$
\log _{4}(64)=3
$$

b. $\quad \log _{6}\left(\frac{1}{36}\right)$

Solution Write the logarithm as

$$
\log _{6}\left(\frac{1}{36}\right)=?
$$

In exponential form this becomes

$$
6^{?}=\frac{1}{36}
$$

We know that $6^{2}=36$ and negative powers lead to reciprocals so $6^{-2}=\frac{1}{36}$. The corresponding logarithmic form gives us the value of the logarithm,

$$
\log _{6}\left(\frac{1}{36}\right)=-2
$$

## Practice

1. Evaluate each logarithm without a calculator.
a. $\quad \log _{8}(64)$
b. $\quad \log _{3}\left(\frac{1}{3}\right)$

## Practice

Evaluate $\log _{3}(32)$ using natural logarithms or common logarithms.

Solution If we write $\log _{3}(32)=$ ? in logarithmic form, we get

$$
3^{?}=32
$$

Since 32 is not a power of 3, use the change of base formula with common logarithms to find the values,

$$
\log _{3}(32)=\frac{\log (32)}{\log (3)} \approx \frac{1.5051}{0.4771} \approx 3.155
$$

Where the common logs are calculated on a calculator. Notice that we could have also done this with natural logarithms,

$$
\log _{3}(32)=\frac{\ln (32)}{\ln (3)} \approx \frac{3.4657}{1.0986} \approx 3.155
$$

2. Evaluate $\log _{6}(37)$ using natural logarithms or common logarithms.

Question 3 - How do you solve problems using logarithms?

## Key Terms

## Summary

If a problem contains an exponential or logarithm function and you need to solve for something inside of the exponential function or logarithm function, converting forms may be useful to solve for the unknown.

Suppose $\$ 5000$ is deposited in an account that earns $2 \%$ compound interest that is done annually. In how many years will there be $\$ 6000$ in the account.

This problem requires the use of the compound interest formula,


Let's look at the quantities in the problem statement:
$\$ 5000$ is deposited in an account $\rightarrow P V=5000$
that earns $2 \%$ compound interest that is done annually $\rightarrow i=0.02$
Will there be $\$ 6000$ in the account $\rightarrow F V=6000$
Putting these values into the formula above gives us

$$
6000=5000(1+0.02)^{n}
$$

Divide both sides by 5000 to get the exponential piece by itself:

$$
\frac{6000}{5000}=(1+0.02)^{n}
$$

Now convert to logarithmic form:

$$
\log _{1.02}\left(\frac{6000}{5000}\right)=n
$$

Calculate the logarithm using common logarithms:

$$
\begin{aligned}
\frac{\log \left(\frac{6000}{5000}\right)}{\log (1.02)} & =n \\
\frac{0.0792}{0.0086} & \approx n
\end{aligned}
$$

Using a calculator to do the logs, we get $n \approx 9.21$ years. Notice how this example requires you to convert to logarithm form and evaluate a logarithm with common logarithms.

## Notes

## Guided Example

## Practice

Find the time required for $\$ 5000$ to grow to at least $\$ 9100$ when deposited at $2 \%$ compounded continuously.

Solution Since interest is being compounded continuously, the future value is given by

$$
F V=P V e^{r t}
$$

The future value is $F V=9100$, the present value is $P V=5000$, and the rate is $r=0.02$. Put these values into the formula above to get

$$
9100=5000 e^{0.02 t}
$$

Divide both sides by 5000 to put the equation in exponential form:

$$
\frac{9100}{5000}=e^{0.02 t}
$$

This converts to a logarithmic form,

$$
0.02 t=\ln \left(\frac{9100}{5000}\right)
$$

To solve for $t$, divide both sides by 0.02 ,

$$
t=\frac{\ln \left(\frac{9100}{5000}\right)}{0.02} \approx 29.94
$$

In approximately 29.94 years, the $\$ 5000$ will have grown to $\$ 9100$.

1. Find the time required for $\$ 2000$ to double when deposited at $8 \%$ compounded continuously. This time is called the doubling time.

## Guided Example

Monthly sales of a Blue Ray player are approximately

$$
S(t)=1000-750 e^{-t} \text { thousand units }
$$

where $t$ is the number of months the Blue Ray player has been on the market.
a. Find the initial sales.

Solution The initial sales occur at $t=0$. The corresponding sales are

$$
S(0)=1000-750 e^{-0}=250 \text { thousand units }
$$

or 250,000 units.
b. In how many months will sales reach 500,000 units?

Solution Set $S(t)$ equal to 500 and solve for $t$.

$$
\begin{array}{rlrl}
500 & =1000-750 e^{-t} & & \\
-500 & =-750 e^{-t} & & \text { Subtract } 1000 \text { from both sides } \\
\frac{-500}{-750} & =e^{-t} & & \text { Divide both sides by }-750 \\
\frac{2}{3} & =e^{-t} & & \text { Reduce the fraction } \\
-t & =\ln \left(\frac{2}{3}\right) & & \text { Convert the exponential form to logarithm form } \\
t & =-\ln \left(\frac{2}{3}\right) \approx 0.41 \text { months } & & \begin{array}{l}
\text { Multiply both sides by }-1 \text { and evaluate the }
\end{array} \\
& \text { logarithm }
\end{array}
$$

c. Will sales ever reach 1000 thousand units?

Solution Follow steps similar to part b.

$$
\begin{array}{rlrl}
1000 & =1000-750 e^{-t} & \text { Set } S(t) \text { equal to } 1000 \\
0 & =-750 e^{-t} & & \text { Subtract } 1000 \text { from both sides } \\
0 & =e^{-t} & \text { Divide both sides by }-750 \\
-t & =\ln (0) & \text { Convert the exponential form to logarithm form }
\end{array}
$$

Since the logarithm of zero is not defined, sales will never be 1000 thousand units.
d. Is there a limit for sales?

To help us answer this question, let's look at a graph of $S(t)$.


Examining the graph, it appears that the sales are getting closer and closer to 1000 units, but never quite get there (part c). So, the limit for sales is 1000 thousand units or 1,000,000 units.

## Practice

2. Monthly sales of a package of mixed nuts are approximately

$$
S(t)=2000-1500 e^{-t} \text { thousand units }
$$

where $t$ is the number of years the mixed nuts has been on the market.
a. Find the initial sales.
b. In how many years will sales reach $1,000,000$ units?
c. Will sales ever reach 2100 thousand units?
d. Is there a limit for sales?

## Section 5.3 Annuities

Question 1 - What is an ordinary annuity?
Question 2 - What is an annuity due?
Question 3 - What is a sinking fund?

Question 1 - What is an ordinary annuity?

## Key Terms

Ordinary annuity

## Summary

A sequence of payments or withdrawals made to or from an account at regular time intervals is called an annuity. The term of the annuity is length of time over which the payments or withdrawals are made. There are several different types of annuities. An annuity whose term is fixed is called an annuity certain. An annuity that begins at a definite date but extends indefinitely is called a perpetuity. If an annuities term is not fixed, it is called a contingent annuity.

The payments for an annuity may be made at the beginning or end of the payment period. In an ordinary annuity, the payments are made at the end of the payment period. An annuity in which the payment period coincides with the interest conversion period is called a simple annuity.

If equal payments of PMT are made into an ordinary annuity for $n$ periods at an interest rate of $i$ per period, the future value of the annuity $F V$ is

$$
F V=\operatorname{PMT}\left[\frac{(1+i)^{n}-1}{i}\right]
$$

## Notes

## Guided Example

Practice
An investor deposits $\$ 1500$ in a simple annuity at the end of each month. This annuity earns $8 \%$ per year, compounded monthly.
a. Find the future value if payments are made for ten years.

Solution Use the ordinary annuity formula

$$
F V=\operatorname{PMT}\left[\frac{(1+i)^{n}-1}{i}\right]
$$

with payment $P M T=1500$, interest rate per period $i=\frac{0.08}{12}$, and the number of periods $n=12 \cdot 10$ or 120 . With these values, the future value is

$$
F V=1500\left[\frac{\left(1+\frac{0.08}{12}\right)^{120}-1}{\frac{0.08}{12}}\right] \approx 274,419.05
$$



The total amount of the 120 payments is $\$ 180,000$ so the annuity has earned $\$ 274,419.05$ $\$ 180,000$ or $\$ 94,419.05$ in interest.
b. What would the investor need to pay each month to accumulate $\$ 500,000$ in ten years?

Solution In this part, we are given the future value $F V=500000$, interest rate per period $i=\frac{0.08}{12}$, and the number of periods $n=12 \cdot 10$ or 120. Put these values into the ordinary annuity formula,

$$
500000=P M T\left[\frac{\left(1+\frac{0.08}{12}\right)^{120}-1}{\frac{0.08}{12}}\right]
$$

1. An employee deposits $\$ 200$ in a simple annuity at the end of each two week pay period. This annuity earns $10 \%$ per year, compounded biweekly.
a. Find the future value if payments are made for thirty years.
b. What would the employee need to pay each pay period to accumulate $\$ 1,000,000$ in thirty years?

To get the payment PMT we need to divide both sides by the quantity in brackets,

$$
\frac{500000}{\left[\frac{\left(1+\frac{0.08}{12}\right)^{120}-1}{\frac{0.08}{12}}\right]}=P M T
$$

We can do this on the calculator by carrying out the calculation below.


Each payment would need to be approximately 2733.05.

## Guided Example

An employee's retirement account currently has a balance of $\$ 100,000$. Suppose the employee contributes $\$ 500$ at the end of each month. If the account earns a return of $5 \%$ compounded monthly, what will the future value of the account in 15 years?

Solution To find the future value of this situation, we need to break it into two parts. In the first part, $\$ 100,000$ grows with compound interest of $5 \%$ compounded monthly for 15 years. In the second part, the employee deposits $\$ 500$ each month into an ordinary annuity that earns $5 \%$ compounded monthly for 15 years. The future value will be the sum of these pieces,

$$
\begin{aligned}
F V & =100000\left(1+\frac{0.05}{12}\right)^{180}+500\left[\frac{\left(1+\frac{0.05}{12}\right)^{180}-1}{\frac{0.05}{12}}\right] \\
& \approx 211370.39+133644.47 \\
& \approx 345014.86
\end{aligned}
$$

The future value will be $\$ 345,014.86$.

## Practice

2. An employee's retirement account currently has a balance of $\$ 50,000$. Suppose the employee contributes $\$ 500$ at the end of each month. If the account earns a return of $6 \%$ compounded monthly, what will the future value of the account in 10 years?

Question 2 - What is an annuity due?

## Key Terms

Annuity due

## Summary

In an annuity due, payments are made at the beginning of the period instead of the end of the period. If equal payments of PMT are made into an annuity due for $n$ periods at an interest rate of $i$ per period, the future value of the annuity $F V$ is

$$
F V=\mathrm{PMT}\left[\frac{(1+i)^{n+1}-1}{i}\right]-\mathrm{PMT}
$$

For an annuity due, an extra period of interest is earned (the $n+1$ in the power) and there is no payment at the end that earns no interest (so PMT is subtracted).

Notes

## Guided Example

## Practice

An investor deposits $\$ 500$ in a simple annuity at the beginning of each quarter. This annuity earns $2 \%$ per year, compounded quarterly. Find the future value if payments are made for five years.

Solution Use the annuity due formula,

$$
F V=\mathrm{PMT}\left[\frac{(1+i)^{n+1}-1}{i}\right]-\mathrm{PMT}
$$

with payment $P M T=500$, interest rate per period $i=\frac{0.02}{4}$, and number of periods $n=4.5$ or 20. Put the values in to give,

$$
F V=500\left[\frac{\left(1+\frac{0.02}{4}\right)^{20+1}-1}{\frac{0.02}{4}}\right]-500 \approx 10542.01
$$

Since 20 deposits of $\$ 500$ each would make the total payments $\$ 10,000$, the annuity earns $\$ 10,542.01-\$ 10,000$ or $\$ 542.01$ in interest.

1. Suppose you deposits $\$ 1000$ in a simple annuity at the beginning of every six months. This annuity earns $1 \%$ per year, compounded semiannually. Find the future value if payments are made for ten years.

Question 3 - What is a sinking fund?

## Key Terms

Sinking fund

## Summary

Annuities that are created to fund a purchase at a later date like some equipment or a college education are called sinking funds. In a sinking fund, the future value is known and another quantity in the annuity formula is being solved for.

Notes

## Guided Example

## Practice

Suppose you want to accumulate $\$ 2,000,000$ in a retirement account in 40 years. The retirement account averages an interest rate of $8 \%$ per year. How much would you need to deposit every two weeks (directly from your paycheck) to accumulate $\$ 2,000,000$ ?

Solution Since deposits are being made at the end of each two week period, this is an ordinary annuity where the future value is $F V=2000000$, the interest rate per period is $i=\frac{0.08}{26}$, and the number of periods is $n=26 \cdot 40$ or 1040 . Put the values into the ordinary annuity formula,

$$
2000000=P M T\left[\frac{\left(1+\frac{0.08}{26}\right)^{1040}-1}{\frac{0.08}{26}}\right]
$$

and solve for the payment $P M T$ :

$$
P M T=\frac{2000000}{\left[\frac{\left(1+\frac{0.08}{26}\right)^{1040}-1}{\frac{0.08}{26}}\right]} \approx 262.85
$$

Each payment would need to be approximately $\$ 262.85$ to accumulate $\$ 2,000,000$.

1. Suppose you want to have $\$ 25,000$ in an account in 6 years to purchase a new vehicle. The account earns $3.25 \%$ per year. How much would you need to put into the account at the end of each month to accumulate $\$ 25,000$ ?

## Section 5.4 Amortization

Question 1 - How do you find the present value of an annuity?
Question 2 - How is a loan amortized?
Question 3 - How do you make an amortization table?

Question 1 - How do you find the present value of an annuity?

## Key Terms

Present value

## Summary

For an ordinary annuity whose present value is PV , the future value is

$$
\mathrm{FV}=\mathrm{PV}(1+i)^{n}+\mathrm{PMT}\left[\frac{(1+i)^{n}-1}{i}\right]
$$

if the payments PMT are made into the annuity which earns interest per period $i$ over $n$ periods. Since the payments are made into the annuity, the second term is added. The future value of the annuity increases.

If the payments are made from the annuity, the second term is subtracted to give

$$
\mathrm{FV}=\mathrm{PV}(1+i)^{n}-\mathrm{PMT}\left[\frac{(1+i)^{n}-1}{i}\right]
$$

In this case, the future value of the annuity decreases since money is removed from the annuity. In some applications, we wish to find the present value (what must be in the account today) so that the account ends up with some amount in the future. To find the percent value, we need to substitute values for FV, i, PMT, and $n$ and solve the resulting equation for PV.

## Notes

## Guided Example

Find the present value of an ordinary annuity with payments of $\$ 10,000$ paid semiannually for 15 years at $5 \%$ compounded semiannually.

Solution We'll use the formula

$$
\mathrm{FV}=\mathrm{PV}(1+i)^{n}-\mathrm{PMT}\left[\frac{(1+i)^{n}-1}{i}\right]
$$

to find the present value PV. Think of this as a decreasing annuity problem where we want the future value to be zero. From the problem statement, we know that

$$
\begin{aligned}
\mathrm{PMT} & =10000 \\
i & =\frac{0.05}{2}=0.025 \\
n & =15 \cdot 2=30
\end{aligned}
$$

Put these values into the formula and solve for PV:

$$
\begin{aligned}
& 0=\operatorname{PV}(1+0.025)^{30}-10000\left[\frac{(1+0.025)^{30}-1}{0.025}\right] \\
& 10000\left[\frac{(1+0.025)^{30}-1}{0.025}\right]=\operatorname{PV}(1+0.025)^{30} \quad \text { Move the second term to the left side } \\
& \frac{10000\left[\frac{(1+0.025)^{30}-1}{0.025}\right]}{(1+0.025)^{30}}=\mathrm{PV}
\end{aligned}
$$

If we evaluate this in a graphing calculator, we get approximately $\$ 209,302.93$. This means that if we deposit $\$ 209,302.93$ today, we can make semiannual payments of $\$ 10,00$ from the annuity for 15 years before there is nothing left in the annuity.

## Practice

1. Find the present value of an ordinary annuity with payments of $\$ 90,000$ paid annually for 25 years at $8 \%$ compounded annually.

Question 2 - How is a loan amortized?

## Key Terms

Payment Amortization

## Summary

Decreasing annuities may be used in auto or home loans. In these types of loans, some amount of money is borrowed. Fixed payments are made to pay off the loan as well as any accrued interest. This process is called amortization.

In the language of finance, a loan is said to be amortized if the amount of the loan and interest are paid using fixed regular payments. From the perspective of the lender, this type of loan is a decreasing annuity. The amount of the loan is the present value of the annuity. The payments from the annuity (to the lender) reduce the value of the annuity until the future value is zero.

Suppose a loan of PV dollars is amortized by periodic payments of PMT at the end of each period. If the loan has an interest rate of $i$ per period over $n$ periods, the payment is

$$
\mathrm{PMT}=\frac{i \mathrm{PV}}{1-(1+i)^{-n}}
$$

Suppose you want to borrow $\$ 10,000$ for an automobile. Navy Federal Credit Union offers a loan at an annual rate of $1.79 \%$ amortized over 12 months. To find the payment, identify the key quantities in the formula:

$$
\begin{aligned}
i & =\frac{0.0179}{12} \\
P V & =10,000 \\
n & =12
\end{aligned}
$$

Put these values into the payment formula to get
$\mathrm{PMT}=\frac{i \mathrm{PV}}{1-(1+i)^{-n}}=\frac{\frac{0.0179}{12} \cdot 10000}{1-\left(1+\frac{0.0179}{12}\right)^{-12}} \approx 841.44$


Pay careful attention to how problems must be rounded. Rounding up, down, or to the nearest cent can change answers drastically.

Notes

## Guided Example

## Practice

Find the payment necessary to amortize a loan of $\$ 7400$ at an interest rate of $6.2 \%$ compounded semiannually in 18 semiannual payments.

Solution To find the payment, use the formula

$$
\mathrm{PMT}=\frac{i \mathrm{PV}}{1-(1+i)^{-n}}
$$

In this case,

$$
\begin{aligned}
\mathrm{PV} & =7400 \\
i & =\frac{0.062}{2}=0.031 \\
n & =18
\end{aligned}
$$

Put the values in the formula to give

$$
\mathrm{PMT}=\frac{0.031 \cdot 7400}{1-(1+0.031)^{-18}} \approx 542.60
$$

To find the total payments, multiply the amount of each payment by 18 to get

$$
542.60(18)=9766.80
$$

To find the total amount of interest paid, subtract the original loan amount from the total payments,

$$
9766.80-7400=2366.80
$$

1. Find the payment necessary to amortize a loan of $\$ 25,000$ at an interest rate of $8.4 \%$ compounded quarterly in 24 quarterly payments.

Question 3 - How do you make an amortization table?
Key Terms
Amortization table

## Summary

An amortization table (also called an amortization schedule) records the portion of the payment that applies to the principal and the portion that applies to interest. Using this information, we can determine exactly how much is owed on the loan at the end of any period.

The amortization table generally has 5 columns and rows corresponding to the initial loan amount and the payments. The heading for each column are shown below.

| Payment <br> Number | Amount of <br> Payment | Interest in <br> Payment | Amount in <br> Payment <br> Applied to <br> Balance | Outstanding <br> Balance at the <br> End of the <br> Period |
| :---: | :---: | :---: | :---: | :---: |

To fill out the table, you need to carry out a sequence of steps to get each row of the table.

1. The first row of the table corresponds to the initial loan balance. Call this payment 0 and place the amount loaned in the column title "Outstanding Balance at the End of the Period".
2. Go to the next line in the table and enter the payment calculated on the loan.
3. In the same row, use $I=P V r t$ to find the interest on the outstanding balance. Place this under the column titled "Interest in Payment".
4. To find the "Amount in Payment Applied to Balance", subtract the "Interest in the Payment" from the "Amount of Payment".
5. To find the new "Outstanding Balance at the End of the Period", subtract the "Amount in Payment Applied to Balance" from the "Outstanding Balance at the End of the Period" in the previous payment.

Fill out these quantities for all payments until the past payment. In the last payment, start by paying off the loan by making "Amount in Payment Applied to Balance" equal to the "Outstanding Balance at the End of the Period" in the second to last payment. This means the loan will be paid off resulting in the "Outstanding Balance at the End of the Period" for the final payment being 0 . Finally, calculate the interest in the final payment and add it to the "Amount in Payment Applied to Balance" to give the final payment. Because of rounding in the payment, this may be slightly higher of lower than the other payments.

Let's look at an example of a $\$ 10,000$ for an automobile. Navy Federal Credit Union offers a loan at an annual rate of $1.79 \%$ amortized over 12 months. The amortization table below shows the calculation of the quantities for payment 1 and the last payment. Other payments follow a similar process.


Suppose a loan of $\$ 2500$ is made to an individual at $6 \%$ interest compounded quarterly. The loan is repaid in 6 quarterly payments.
a. Find the payment necessary to amortize the loan.

Solution To find the payment on the loan, use the formula

$$
P M T=\frac{i \cdot P V}{1-(1+i)^{-n}}
$$

For this problem, the interest rate per period is $i=\frac{0.06}{4}$. The present value is $P V=2500$ and the number of periods is $n=6$. Using these values gives

$$
P M T=\frac{\frac{0.06}{4} \cdot 2500}{1-\left(1+\frac{0.06}{4}\right)^{-6}} \approx 438.813
$$

Depending on how the rounding is done, this gives a payment of $\$ 438.81$ or 438.82 . For a calculated payment, the payment is often rounded to the nearest penny. However, many finance companies will round up to insure the last payment is no more than the other payments.
b. Find the total payments and the total amount of interest paid based on the calculated monthly payments.

Solution The total payments (assuming the payment is rounded to the nearest penny) is

$$
438.81(6)=2632.86
$$

The total amount of interest is

$$
2632.86-2500=132.86
$$

c. Find the total payments and the total amount of interest paid based on an amortization table.

Solution Making the amortization table takes several steps. Let me take it in pieces using the payment from above.

| Payment <br> Number | Amount of Payment | Interest in Payment | Amount in Payment Applied to Balance | Outstanding Balance at the End of the Period |
| :---: | :---: | :---: | :---: | :---: |
| 0 |  |  |  | 2500 |
| 1 | 438.81 | 37.50 | 401.31 | 2098.69 |
|  |  |  |  |  |
|  |  |  |  |  |

The next row is filled out in a similar manner.

| Payment <br> Number | Amount of <br> Payment | Interest in <br> Payment | Amount in <br> Payment <br> Applied to <br> Balance | Outstanding <br> Balance at the <br> End of the <br> Period |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 |  |  |  | 2500 |  |
| 1 | 438.81 | 37.50 | 401.31 | 2098.69 |  |
| 2 | 438.81 | 31.48 | 407.33 | 1691.36 |  |
| 个 |  |  |  |  |  |
|  | $\frac{0.06}{4} \cdot 2098.69$ | $438.81-31.48$ | $2098.69-407.33$ |  |  |

Continue this process until the last row

| Payment <br> Number | Amount of <br> Payment | Interest in <br> Payment | Amount in <br> Payment <br> Applied to <br> Balance | Outstanding <br> Balance at the <br> End of the <br> Period |
| :---: | :---: | :---: | :---: | :---: |
| 0 |  |  |  | 2500 |
| 1 | 438.81 | 37.50 | 401.31 | 2098.69 |
| 2 | 438.81 | 31.48 | 407.33 | 1691.36 |
| 3 | 438.81 | 25.37 | 413.44 | 1277.92 |
| 4 | 438.81 | 19.17 | 419.64 | 850.28 |
| 5 | 438.81 | 12.87 | 425.94 | 432.34 |
| 6 |  |  |  |  |

After the fifth payment, we have $\$ 432.34$ of principal left to pay in the final payment. So, this is the principal in the sixth payment. The interest is found by paying interest on the outstanding balance,

$$
\frac{0.06}{4} \cdot 432.34 \approx 6.49
$$

This gives a final payment of

$$
432.34+6.49=438.83
$$

Now put these numbers into the amortization table.

| Payment <br> Number | Amount of <br> Payment | Interest in <br> Payment | Amount in <br> Payment <br> Applied to <br> Balance | Outstanding <br> Balance at the <br> End of the <br> Period |
| :---: | :---: | :---: | :---: | :---: |
| 0 |  |  |  | 2500 |
| 1 | 438.81 | 37.50 | 401.31 | 2098.69 |
| 2 | 438.81 | 31.48 | 407.33 | 1691.36 |
| 3 | 438.81 | 25.37 | 413.44 | 1277.92 |
| 4 | 438.81 | 19.17 | 419.64 | 850.28 |
| 5 | 438.81 | 12.87 | 425.94 | 432.34 |
| 6 | 438.83 | 6.49 | 432.34 | 0 |

Since the payments had been rounded to the nearest penny (rounded down), the final payment is slightly higher than the previous payments. Adding all of the payments we get a total of $\$ 2632.88$. Adding the interest amounts gives total interest of $\$ 132.88$.

## Practice

1. Suppose a loan of $\$ 5000$ is made to an individual at $4 \%$ interest compounded semiannually. The loan is repaid in 6 semiannual payments.
a. Find the payment necessary to amortize the loan. Round the payment to the nearest penny.
b. Find the total payments and the total amount of interest paid based on the calculated monthly payments.
c. Find the total payments and the total amount of interest paid based on an amortization table.

## Chapter 5 Solutions

## Section 5.1

Question 1 1) $\$ 80,2$ ) $\$ 18,415,3$ ) approximately $2.4 \%$
Question 2 1) $\$ 103,507.98$, 2) $8.96 \%$, 3) $\$ 12561.52$
Question 31 ) approximately $2.22 \%$
Question 4 1) $\$ 5684.53,2) \$ 10,025.88$

## Section 5.2

Question $1 \quad$ 1a) $\log _{5}(125)=3,1$ b) $\log _{2}\left(\frac{1}{16}\right)=-4,1$ c) $\ln \left(\frac{F V}{P V}\right)=r t$
2a) $7^{2}=49,2$ b) $10^{-3}=0.001,2$ c) $10^{Q}=1+Z$
$\begin{array}{ll}\text { Question } 2 & \text { 1a) } 2,1 \mathrm{~b})-1\end{array}$
2) approximately 2.0153

Question 3 1) approximately 8.66 years
2a) 500 thousand units, 2b) approximately 0.405 years, 2c) no, 2d) highest sales get is 2000 thousand units

Section 5.3
Question $1 \quad$ 1a) $\$ 986,454.92,1 b) \$ 202,75,2) \$ 172,909.51$
Question 2 1) \$21,084.01
Question 3 1) \$314.94

## Section 5.4

Question 1 1) \$960,729.86
Question 2 1) $\$ 1336.80$
Question 3) 1a) $\$ 892.63$, 1b) $\$ 5355.78, \$ 355.78$,

1c)

| Payment <br> Number | Amount of <br> Payment | Interest in <br> the <br> Payment | Amount in <br> Payment <br> Applied to <br> Balance | Outstanding <br> Balance at <br> the End of <br> the Period |
| ---: | ---: | ---: | ---: | ---: |
| 0 |  |  |  | 5000 |
| 1 | 892.63 | 100.00 | 792.63 | 4207.37 |
| 2 | 892.63 | 84.15 | 808.48 | 3398.89 |
| 3 | 892.63 | 67.98 | 824.65 | 2574.24 |
| 4 | 892.63 | 51.48 | 841.15 | 1733.09 |
| 5 | 892.63 | 34.66 | 857.97 | 875.12 |
| Total | 892.62 | 17.50 | 875.12 | 0.00 |
|  | 5355.77 | $\mathbf{3 5 5 . 7 7}$ |  |  |

