Suppose that the cost function for a product is given by $C(x) = .002x^3 - 9x + 4000$. Find the production level that will produce the minimum average cost per unit $\overline{C}(x)$.

To do this problem, we need to understand what is meant by average cost per unit. The average cost per unit is defined as

$$\overline{C}(x) = \frac{C(x)}{x}$$

You can think of it as the cost divide by the total number of units x at that cost.

Find the average cost function: Divide the given cost function by x and simplify to yield

$$\begin{aligned}\overline{C}(x) &= \frac{.002x^3 - 9x + 4000}{x} \\ &= \frac{.002x^3}{x} - \frac{9x}{x} + \frac{4000}{x} \\ &= .002x^2 - 9 + \frac{4000}{x}\end{aligned}$$

Find the critical point of the average cost function: To take the derivative of the average cost function, rewrite it as

$$\overline{C}(x) = .002x^2 - 9 + 4000x^{-1}$$

Using the power rule, we get a derivative of

$$\begin{aligned}\overline{C}'(x) &= .004x - 4000x^{-2} \\ &= .004x - \frac{4000}{x^2}\end{aligned}$$

This function is undefined at x = 0, and is equal to zero when

$$.004x - \frac{4000}{x^2} = 0$$

If we clear the fraction by multiplying each term by x^2 we get

$$.004x^3 - 4000 = 0$$

Now isolate x^3 and take the cube root of both sides:

$$
\begin{aligned}
.004x^3 &= 4000 \\
x^3 &= 1000000 \\
x &= \sqrt[3]{1000000} \\
x &= 100
\end{aligned}
$$

So we have critical points at x = 0 and x = 100. Of course, it doesn't make sense to find an average cost over 0 units, so we only need consider x = 100.

Prove that the critical point is a minimum: We could make a number line and test on either side of the critical point (first derivative point), but in this problem it is easier to evaluate the second derivative at the critical point. The second derivative is

$$
\begin{aligned}
\overline{C}''(x) &= .004 + 8000x^{-3} \\
&= .004 + \frac{8000}{x^3}
\end{aligned}
$$

Putting in the critical point,

$$\overline{C}''(100) = .004 + \frac{8000}{100^3} > 0$$

At the critical point the graph is concave up so the critical point must be a minimum.

Find the minimum average cost at the critical point: If we put the critical point in the average cost function, we find the cheapest that we can manufacture per unit:

$$\overline{C}(100) = .002(100)^2 - 9 + \frac{4000}{100} = 51$$

This means that if we manufacture 100 units, the cost per unit is $51 per unit and that is the lowest cost per unit possible. We can verify this by viewing a graph of $\overline{C}(x)$: