An animal food must provide at least 54 units of vitamins and 60 calories per serving. One gram of soybean meal provides 2.5 units of vitamins and 5 calories. One gram of meat byproducts provides 4.5 units of vitamins and 3 calories. One gram of grain provides 5 units of vitamins and 10 calories. A gram of soybean meal costs 8¢, a gram of meat byproducts 9¢, and a gram of grain 10¢. What mixture of these three ingredients will provide the required vitamins and calories at minimum cost?

Let's attack this in pieces.

Find the objective function: Since the problem says to minimize the cost, so we'll start out by writing

Minimize C =

To go any farther, we need to know what out variables will be. Notice the phrase "*What mixture of these three ingredients"* This leads me to write

y_1: amount of soybean meal in grams
y_2: amount of meat byproducts in grams
y_3: amount of grain in grams

To finish the objective function, I need to know the relationship between the variables and the cost. This is connected through the phrase "*A gram of soybean meal costs 8¢, a gram of meat byproducts 9¢, and a gram of grain 10¢".* Using this information I write

Minimize $C = 8y_1 + 9y_2 + 10y_3$

Find the constraints: Production is constrained by two phrases, "*must provide at least 54 units of vitamins" and "must provide at least 60 calories per serving".* This leads me to write

total amount of vitamins $\geq$ 54 units of vitamins

total number of calories $\geq$ 60 calories per serving

How do the variables relate to vitamins and calories? Start by looking at

One gram of soybean meal provides 2.5 units of vitamins
One gram of meat byproducts provides 4.5 units of vitamins → total amount of vitamins = $2.5y_1 + 4.5y_2 + 5y_3$
One gram of grain provides 5 units of vitamins

So our constraint on vitamins is $2.5y_1 + 4.5y_2 + 5y_3 \geq 54$

Applying the same type of reasoning to calories shows

One gram of soybean meal provides 5 calories
One gram of meat byproducts provides 3 calories → total number of calories = $5y_1 + 3y_2 + 10y_3$
One gram of grain provides 10 calories

So our constraint on calories is $5y_1 + 3y_2 + 10y_3 \geq 60$.

Write out the linear programming problem: Now we add the non-negativity constraints and we get

Minimize $C = 8y_1 + 9y_2 + 10y_3$

Subject to

$2.5y_1 + 4.5y_2 + 5y_3 \geq 54$

$5y_1 + 3y_2 + 10y_3 \geq 60$

$y_1 \geq 0,\ y_2 \geq 0,\ y_3 > 0$

Write out the dual problem: Take the coefficients of this problem and write out

$$\left[\begin{array}{ccc|c} 2.5 & 4.5 & 5 & 54 \\ 5 & 3 & 10 & 60 \\ \hline 8 & 9 & 10 & 0 \end{array}\right]$$

Take the transpose of this matrix in switching the rows and columns:

$$\left[\begin{array}{cc|c} 2.5 & 5 & 8 \\ 4.5 & 3 & 9 \\ 5 & 10 & 10 \\ \hline 54 & 60 & 0 \end{array}\right]$$

Now we can use this matrix to write out the dual maximization problem:

Maximize $z = 54x_1 + 60x_2$

Subject to

$2.5x_1 + 5x_2 \leq 8$

$4.5x_1 + 3x_2 \leq 9$

$5x_1 + 10x_2 \leq 10$

$x_1 \geq 0,\ x_2 \geq 0$

Add the slack variables and write out the initial tableau: We'll need three slack variables for each of the three constraints.

$2.5x_1 + 5x_2 + s_1 = 8$

$4.5x_1 + 3x_2 + s_2 = 9$

$5x_1 + 10x_2 + s_3 = 10$

Move all the variables over to left hand side in the objective function gives $-54x_1 - 60x_2 + z = 0$. Now we can write out the initial tableau

$$\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ \end{array}$$
$$\left[\begin{array}{cccccc|c} 2.5 & 5 & 1 & 0 & 0 & 0 & 8 \\ 4.5 & 3 & 0 & 1 & 0 & 0 & 9 \\ 5 & 10 & 0 & 0 & 1 & 0 & 10 \\ \hline -54 & -60 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Find the pivot element: Locate the pivot column by finding the most negative entry in the indicator row, -60. Now form ratios from the last column and the pivot column:

$$\begin{array}{c} \frac{8}{5} \\ \frac{9}{3} \\ \frac{10}{10} \\ \\ \end{array} \left[\begin{array}{cccccc|c} 2.5 & 5 & 1 & 0 & 0 & 0 & 8 \\ 4.5 & 3 & 0 & 1 & 0 & 0 & 9 \\ 5 & \color{red}{10} & 0 & 0 & 1 & 0 & 10 \\ \hline -54 & -60 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

The pivot row is the row with the smallest ratio. The pivot row and pivot column cross at the pivot element, 10.

Use row operations to make the pivot a 1 and the rest of the column zeros: To make the 10 into a 1, multiply the third row by 1/10 (1/10 R3 → R3) to give

$$\left[\begin{array}{cccccc|c} 2.5 & 5 & 1 & 0 & 0 & 0 & 8 \\ 4.5 & 3 & 0 & 1 & 0 & 0 & 9 \\ .5 & 1 & 0 & 0 & .1 & 0 & 1 \\ \hline -54 & -60 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Now use more row operations make the rest of the second column into zeros:

$$\begin{array}{c} -5R3+R1 \to R1 \\ -3R3+R2 \to R2 \\ 60R3+R4 \to R4 \end{array} \Rightarrow \left[\begin{array}{cccccc|c} 0 & 0 & 1 & 0 & -.5 & 0 & 3 \\ 3 & 0 & 0 & 1 & -.3 & 0 & 6 \\ .5 & 1 & 0 & 0 & .1 & 0 & 1 \\ \hline -24 & 0 & 0 & 0 & 0 & 1 & 60 \end{array}\right]$$

Since there is still a negative in the indicator row, the first column is the new pivot column. Form the ratios with this column to yield

$$\begin{array}{c} \\ \frac{6}{3} \\ \frac{1}{.5} \\ \\ \end{array} \left[\begin{array}{cccccc|c} 0 & 0 & 1 & 0 & -.5 & 0 & 3 \\ 3 & 0 & 0 & 1 & -.3 & 0 & 6 \\ .5 & 1 & 0 & 0 & .1 & 0 & 1 \\ \hline -24 & 0 & 0 & 0 & 0 & 1 & 60 \end{array}\right]$$

Ratios with a negative denominator or zero in the denominator should be ignored. The two remaining ratios are both equal to zero. So we'll make 3 the pivot first, then we'll start from the step above and make .5 the pivot (usually we don't have this inconvenience).

$$\begin{array}{l} \frac{1}{3}R2 \to R2 \\ -.5R2 + R3 \to R3 \\ 24R2 + R4 \to R4 \end{array} \Rightarrow \left[\begin{array}{cccccc|c} 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 & 3 \\ 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{10} & 0 & 2 \\ 0 & 1 & 0 & -\frac{1}{6} & \frac{3}{20} & 0 & 0 \\ \hline 0 & 0 & 0 & 8 & \frac{18}{5} & 1 & 108 \end{array}\right]$$

If we take the matrix from the bottom of the previous page and make the .5 the pivot, we get

$$\begin{array}{l} 2R3 \to R3 \\ -3R3 + R2 \to R2 \\ 24R3 + R4 \to R4 \end{array} \Rightarrow \left[\begin{array}{cccccc|c} 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 & 3 \\ 0 & -6 & 0 & 1 & -\frac{9}{10} & 0 & 0 \\ 1 & 2 & 0 & 0 & \frac{1}{5} & 0 & 2 \\ \hline 0 & 48 & 0 & 0 & \frac{24}{5} & 1 & 108 \end{array}\right]$$

Let's look at these two matrices closer since both of them indicate a solution:

$$\begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & s_3 & z \end{array}$$
$$\left[\begin{array}{cccccc|c} 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 & 3 \\ 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{10} & 0 & 2 \\ 0 & 1 & 0 & -\frac{1}{6} & \frac{3}{20} & 0 & 0 \\ \hline 0 & 0 & 0 & 8 & \frac{18}{5} & 1 & 108 \end{array}\right]$$

means $y_1 = 0$, $y_2 = 8$, $y_3 = 18/5$ and C = 108

$$\begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & s_3 & z \end{array}$$
$$\left[\begin{array}{cccccc|c} 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 & 3 \\ 0 & -6 & 0 & 1 & -\frac{9}{10} & 0 & 0 \\ 1 & 2 & 0 & 0 & \frac{1}{5} & 0 & 2 \\ \hline 0 & 48 & 0 & 0 & \frac{24}{5} & 1 & 108 \end{array}\right]$$

means $y_1 = 0$, $y_2 = 0$, $y_3 = 24/5$ and C = 108

where the solution for the minimization problem is found under the slack variable. Both solutions lead to the same minimum value of c = 108. So this problem has two solutions! Depending on the requirements of the company producing the food, we might prefer one solution over another. Or we might produce two different foods based on the two solutions.