

**Question 4: What is the product rule for probability?**

The rule for computing conditional probability can be interpreted differently. In Question 2, we defined the conditional probability  $P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$ . If we multiply both sides of this equation by  $P(B)$ , we get

$$P(A|B)P(B) = P(A \text{ and } B)$$

We can also apply this strategy to the conditional probability  $P(B|A) = \frac{P(B \text{ and } A)}{P(A)}$  to obtain a similar expression,

$$P(B|A)P(A) = P(B \text{ and } A)$$

These expressions give the joint probability of  $A$  and  $B$  as a product of a conditional probability and a marginal probability.

**Product Rule for Probability**

The probability of the event  $A$  and  $B$  is

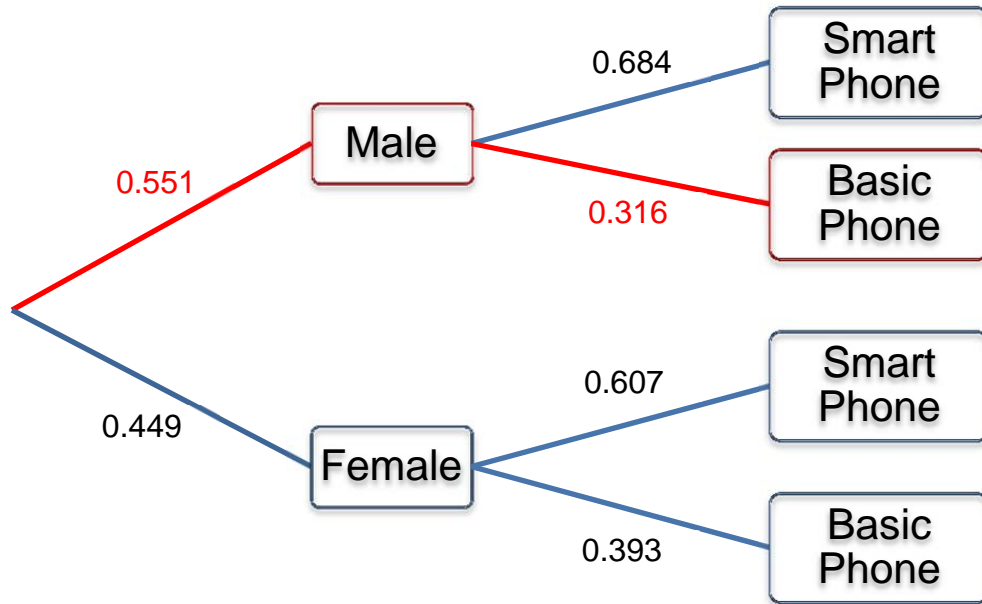
$$P(A \text{ and } B) = P(A|B)P(B)$$

or

$$P(A \text{ and } B) = P(B|A)P(A)$$

Note that we have used the fact that the collection of outcomes in  $A$  and  $B$  is the same as the collection of outcomes in  $B$  and  $A$ .

This rule is very useful when we work with tree diagrams. Let's return to the consumer cell phone survey. We constructed the tree diagram for this survey shown below.



In this tree diagram, a branch passing through the event M, “consumer is male”, and the event B, “consumer owns a basic phone”, is colored red. The first part of the branch ending at male is labeled with the probability 0.551. This is the probability that the consumer is male,  $P(M) = 0.551$ . The part of the branch beginning with male and ending at basic phone is labeled with the probability 0.316. This is the probability that the consumer owns a basic phone given that the consumer is male,  $P(B|M) = 0.316$ . Using the Product Rule for Probability, we may multiply these values to give

$$\underbrace{(0.316)}_{P(B|M)} \underbrace{(0.551)}_{P(M)} \approx \underbrace{0.174}_{P(B \text{ and } M)}$$

In words, the product of the probabilities along the branch gives us the likelihood that a consumer owns a basic phone and is male.

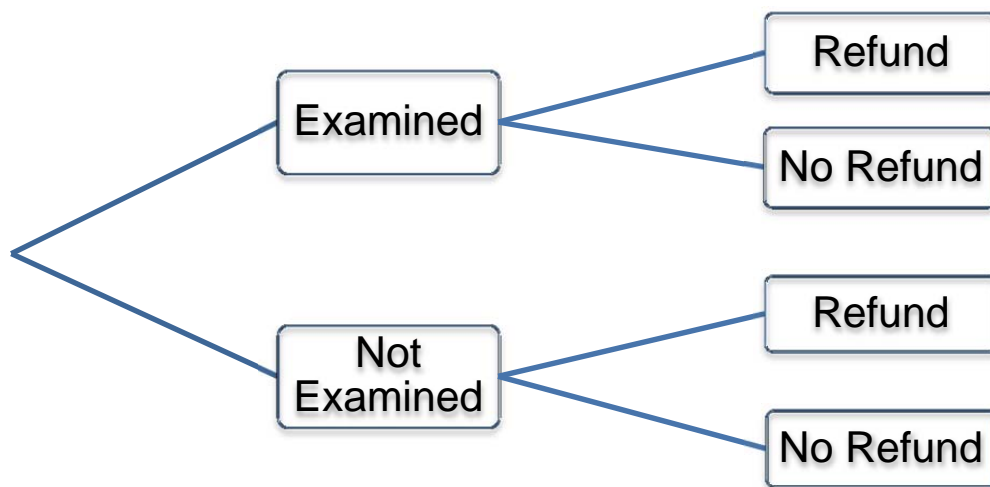
### Product Rule for Tree Diagrams

The product of all probabilities along a branch on a tree diagram is the likelihood of all events occurring that are on the branch.

## Example 5 Product Rule for Tree Diagrams

In 2008, 1.01% of all individual tax returns were selected for further examination. Of those examined, 3.59% received a refund on taxes paid. Of the returns that were not examined, 86.44% received a refund on taxes paid. Use this information to answer the parts below.

- a. Label the branches on the tree diagram below.



**Solution** To make it easier to work with the events, define the events below:

$E$ : return is selected for further examination

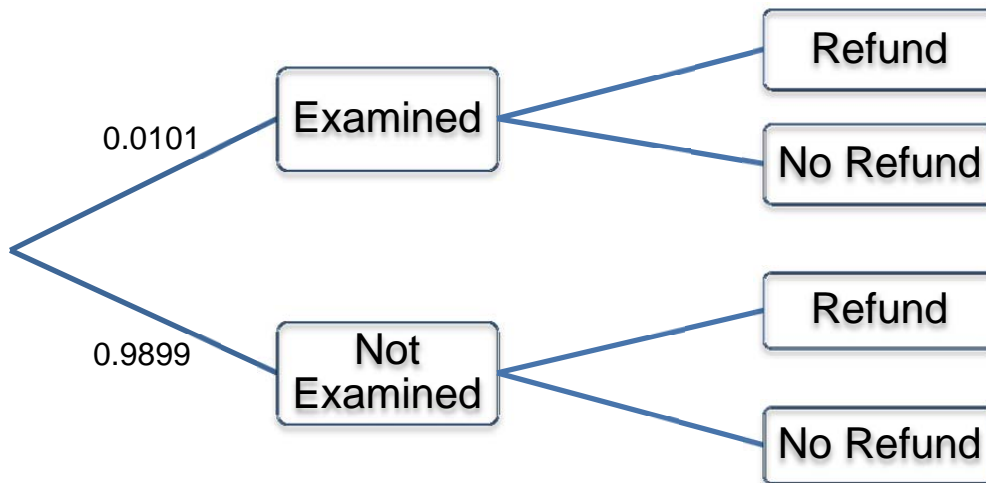
$E'$ : return is not selected for further examination

$R$ : return results in a refund of taxes paid

$R'$ : return does not result in a refund of taxes paid

Using complements allows us to limit the number of letters used to represent the different events. Using these events and the fact that 1.01% of returns are selected for further examination, we write

$P(E) = 0.0101$ . In addition, we can use the fact that the sum of probabilities from any event must be 1 ( $P(E) + P(E') = 1$ ). This tells us that  $P(E') = 0.9899$ . Label these branches on the tree diagram.

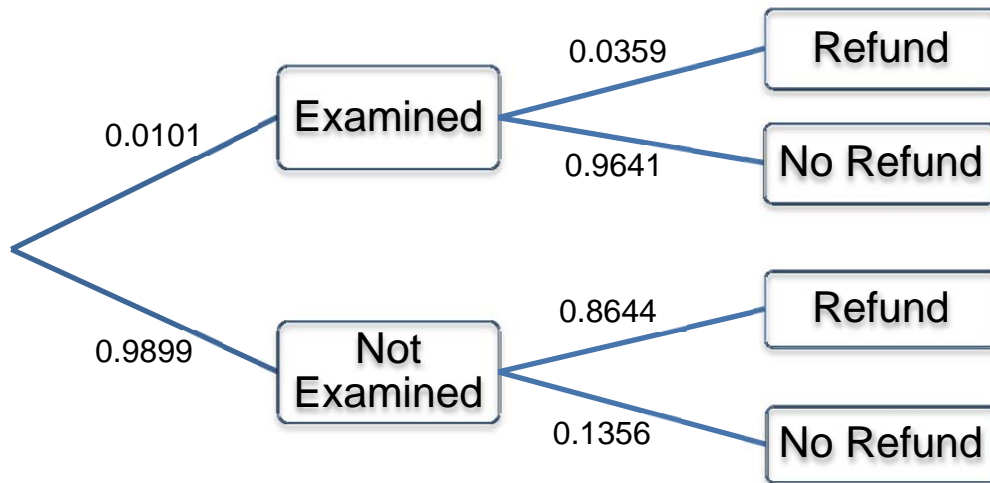


Now let's look at the statement, "Of those examined, 3.59% received a refund on taxes paid." This tells us that  $P(R|E) = 0.0359$ . The other branch originating at  $E$  is the complement, so  $P(R'|E) = 1 - 0.0359 = 0.9641$ . A similar strategy gives us the probabilities originating at  $E'$ ,

$$P(R|E') = 0.8644$$

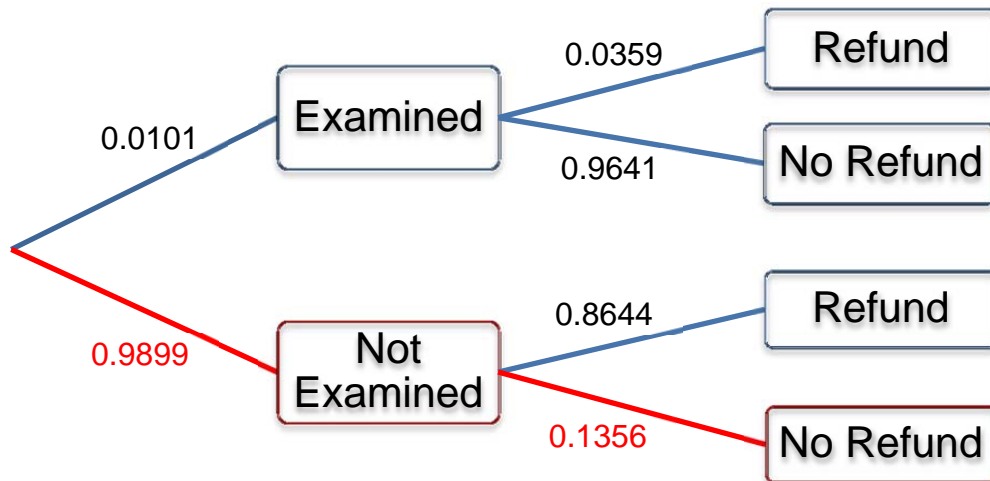
$$P(R'|E') = 1 - 0.8644 = 0.1356$$

This gives the completed tree diagram below.



- b. Find the likelihood that a return is not selected for examination and the return does not result in a refund.

**Solution** The event we are interested in is  $E'$  and  $R'$ . These individual events are highlighted in the tree diagram below.

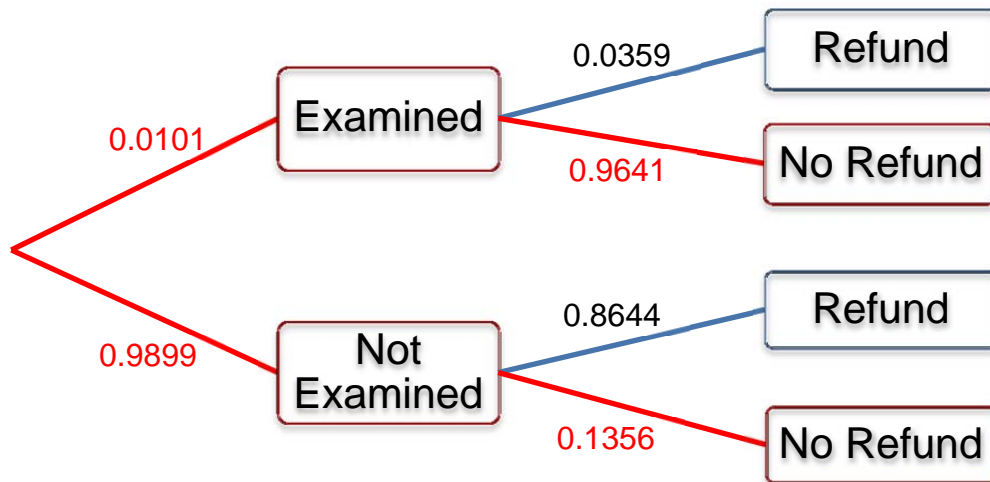


The joint probability is the product of the probabilities along the highlighted branch,

$$P(E' \text{ and } R') = \underbrace{(0.1356)}_{P(R'|E')} \underbrace{(0.9899)}_{P(E')} \approx 0.1342$$

c. Find the probability that a return results in no refund.

**Solution** Two branches along the tree terminate at returns that result in no refund.



As we saw in part b, returns along the lower branch are not examined and result in no refund ( $E'$  and  $R'$ ). Returns along the upper branch are examined and result in no refund ( $E$  and  $R'$ ). Together, these branches give all returns that result in no return. In effect, the returns in  $R'$  are the same as the returns in ( $E'$  and  $R'$ ) or ( $E$  and  $R'$ ). Since there are no returns that correspond to both branches, we can find the likelihood of a return resulting in no refund by applying the product rule to each branch and then add the results:

$$\begin{aligned}
 P(R') &= P((E' \text{ and } R') \text{ or } (E \text{ and } R')) \\
 &= P(E \text{ and } R') + P(E' \text{ and } R') \\
 &= (0.0101)(0.9641) + (0.9899)(0.1356) \\
 &\approx 0.1440
 \end{aligned}$$



In general, there are no outcomes in common to branches corresponding joint probabilities in a tree diagram. This means we may add probabilities from these branches to create other events that are compound events created with “or”. This strategy will be useful for computing other marginal probabilities for Bayes’ Rule in Question 5.