Question 5: How is Bayes' Rule used to compute conditional probability?

In Question 2, we learned that the likelihood of an event A occurring given that an event B has already occurred is

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

We can also use the same basic expression to find the likelihood of an event B occurring given that an event A has already occurred,

$$P(B|A) = \frac{P(B \text{ and } A)}{P(A)}$$

Each of these expressions may be solved for the joint probability in the numerator to give

$$P(A|B)P(B) = P(A \text{ and } B)$$

$$P(B|A)P(A) = P(B \text{ and } A)$$

The joint event A and B is exactly the same event as the joint event B and A. This means their probabilities are also the same. Setting the left sides of these expressions equal gives

$$P(A|B)P(B) = P(B|A)P(A)$$

We can solve for either conditional probability, but if we solve for $P(B|A)$ we get the most basic form of Bayes' Rule.

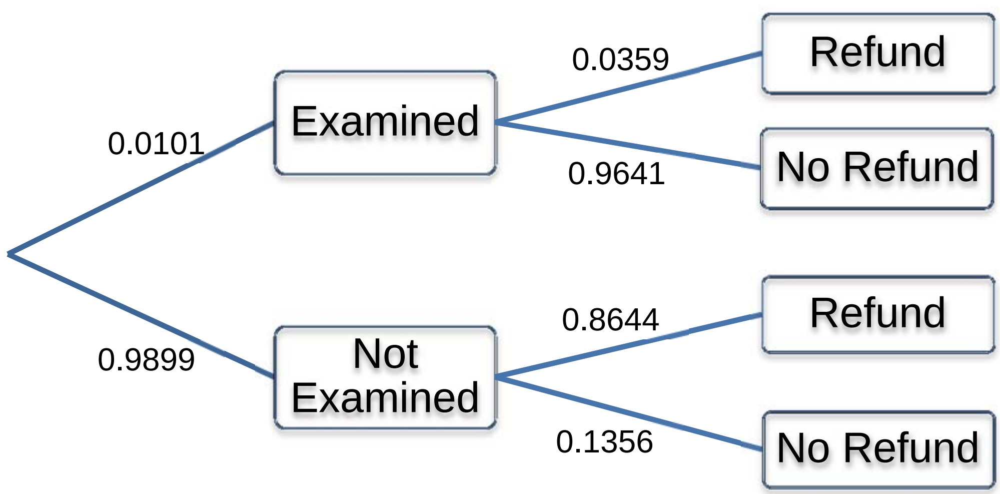
Bayes' Rule

If A and B are events, the conditional probability $P(B|A)$ may be computed in terms of $P(A|B)$ using

$$P(B|A)=\frac{P(A|B)P(B)}{P(A)}$$

This expression allows us to compute one conditional probability in terms of the "reverse" conditional probability. In practice, the most challenging part of using Bayes' Rule is identifying the events and computing the probabilities on the right side. We can simplify this task using a tree diagram.

For instance, let's return to the tax return tree diagram we developed in Question 4.



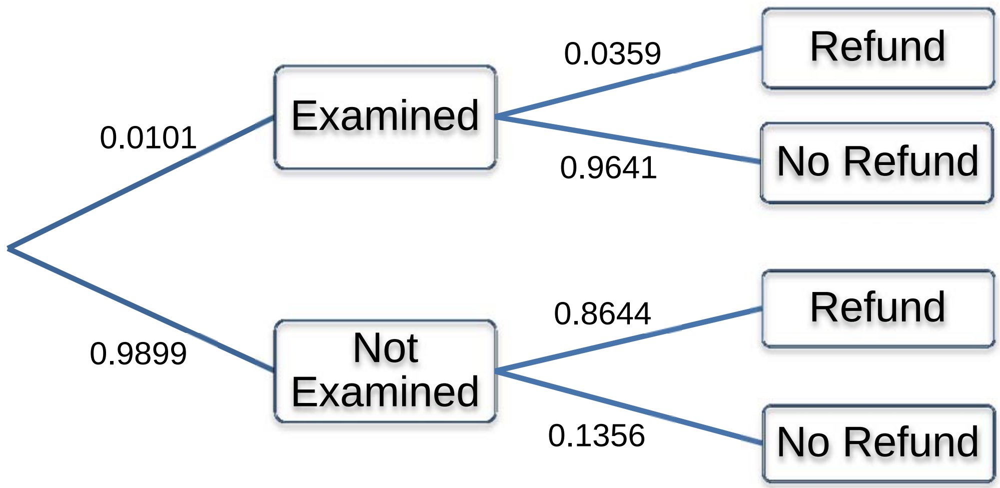
This tree diagram is defined in terms of the marginal probabilities $P(E)$ and $P(E')$, as well as the conditional probabilities $P(R|E)$, $P(R'|E)$, $P(R|E')$, and $P(R'|E')$. If we want to find the likelihood of one of these conditional probabilities reversed such as $P(E|R)$, we apply Bayes' Rule to give

$$P(E \mid R) = \frac{P(R \mid E)P(E)}{P(R)}$$

and use the tree diagram to find the probabilities in the numerator and denominator.

Example 6 Bayes’ Rule

The tree diagram for tax returns is shown below.



Using the events

E: return is selected for further examination

E': return is not selected for further examination

R: return results in a refund of taxes paid

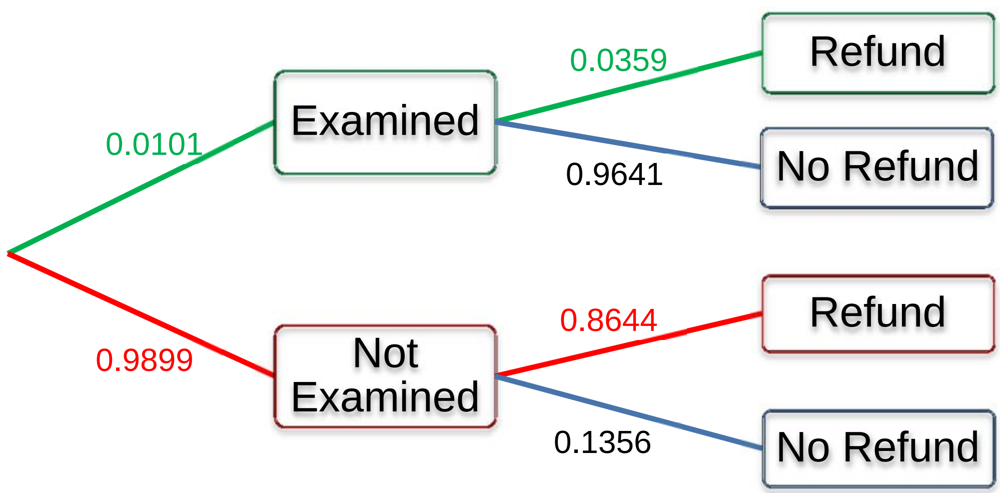
R': return does not result in a refund of taxes paid

find the probability that a return was examined if we already know it resulted in a refund.

Solution We want to find $P(E \mid R)$. Since the tree diagram is drawn to correspond to events given E or E', we'll apply Bayes' Rule to "reverse" the conditional probabilities. For this conditional probability, Bayes' Rule gives us

$$P(E \mid R) = \frac{P(R \mid E)P(E)}{P(R)}$$

We can locate and highlight these probabilities on the tree diagram.



The probabilities in the numerator are located along the green branch. The probability in the denominator is found using the red and green branches which all terminate at R. Put the numbers in to yield

$$\begin{aligned} P(E \mid R) &= \frac{P(R \mid E)P(E)}{P(R)} \\ &= \frac{(0.0359)(0.0101)}{(0.0359)(0.0101)+(0.8644)(0.9899)} \\ &\approx 0.0004 \end{aligned}$$

If a return results in a refund, it is unlikely the return was examined.