The total profit P(x) (in thousands of dollars) from the sale of x units of a certain prescription drug is given by

$$P(x) = \ln\left(-x^3 + 3x^2 + 72x + 1\right)$$

for x in [0, 10]. What sales will lead to maximum profit?

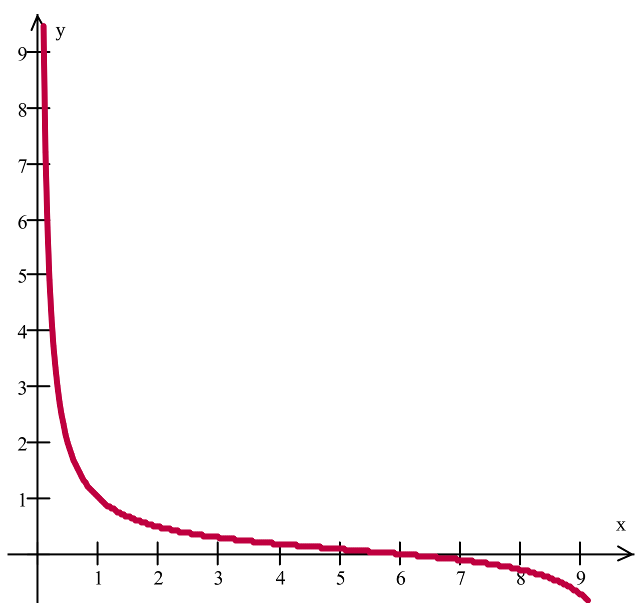
Find the derivative of P(x): The function is a composition, so we'll need the chain rule to find the derivative:

$$\begin{array}{c} inside = -x^3 + 3x^2 + 72x + 1 \\ outside = \ln(x) \end{array} \quad \rightarrow \quad \begin{array}{c} inside' = -3x^2 + 6x + 72 \\ outside' = \tfrac{1}{x} \end{array}$$

Using this information, we get

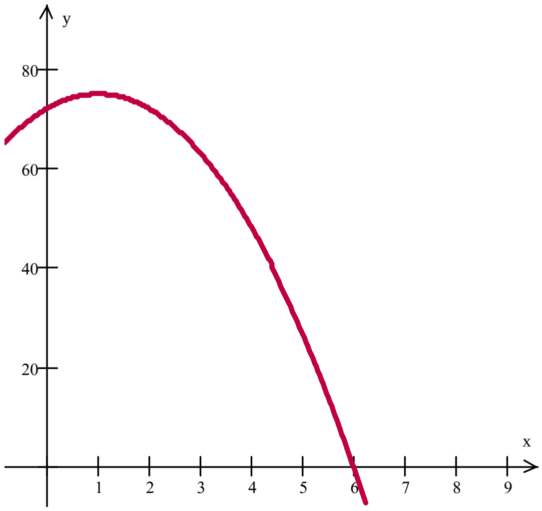
$$P'(x) = \frac{1}{-x^3 + 3x^2 + 72x + 1} \cdot \left(-3x^2 + 6x + 72\right)$$

Find the critical numbers: Since the derivative is a fraction, the critical numbers will be where the derivative is zero or undefined. To get an idea of what is going on, let's examine the derivative over the interval [0, 10].

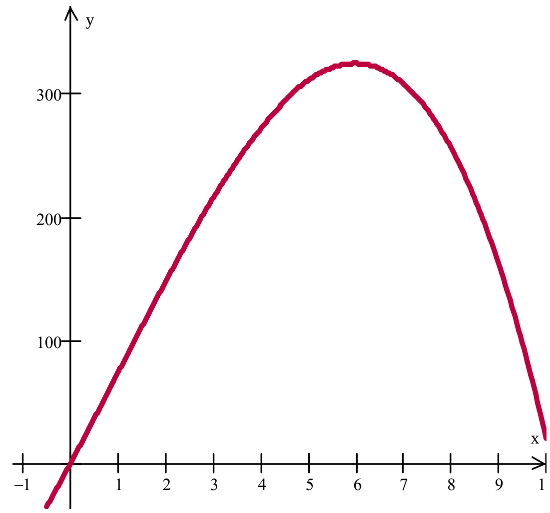


It looks like there is a zero around x = 6 and perhaps an asymptote near x = 0. To find each of these, let's look at the graph of

$y = -3x^2 + 6x + 72$



$y = -x^3 + 3x^2 + 72x + 1$

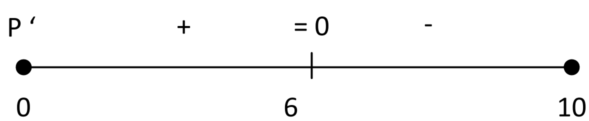


The graph on the left is a graph of the numerator of the derivative. It is equal to zero at x = 6 since the numerator factors as

$$-3x^2 + 6x + 72 = -3\left(x^2 - 2x - 24\right)$$
$$= -3(x-6)(x+4)$$

The numerator is also equal to 0 at x = -4, but that is outside [0, 10]. The denominator is on the right and between 0 and 10, is always positive. So our only critical number on [0, 10] is at x = 6.

Apply the first derivative test: Make a number line from x = 0 to x = 10 and test on either side of the critical number:



Since the function is increasing on the left hand side of x = 6 and decreasing on the other side, the critical point must be a relative maximum. So 6 units of sales leads to a maximum profit of

$$P(6) = \ln\left(-6^3 + 3(6)^2 + 72(6) + 1\right) \approx 5.78383 \text{ thousand dollars}$$

or $5783.83.