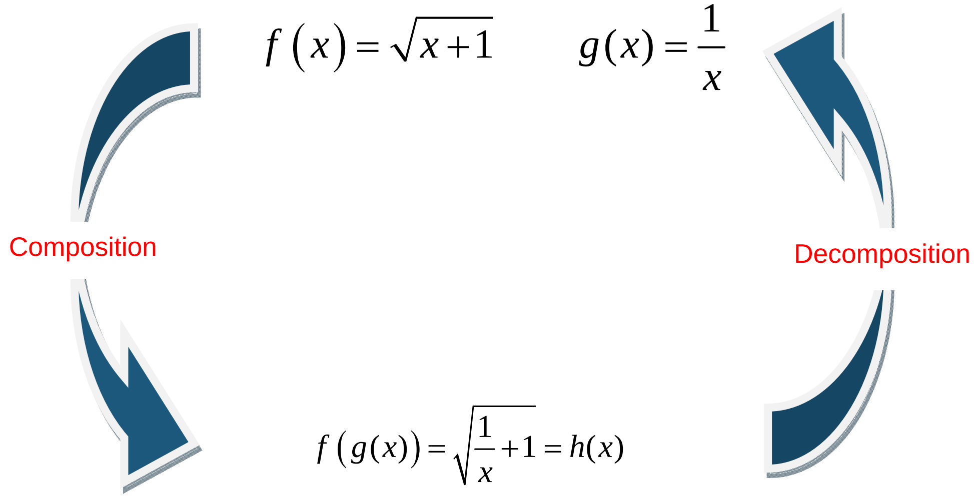
Question 2: How do you write a function as a composition of two functions?

For our purpose, the process of decomposing a function into two functions will be more useful than composition. By decomposing, we mean writing a function as a composition of two other functions. This is the reverse of the process discussed in the previous question.



$$f(x) = \sqrt{x+1} \qquad g(x) = \frac{1}{x}$$

$$f(g(x)) = \sqrt{\frac{1}{x}+1} = h(x)$$

To help us understand how to decompose a function, it is useful to examine what a composition actually does.

Let's look at what the composition $f(g(x)) = \sqrt{\frac{1}{x}+1}$ does to an input x. If we start with a value for x, several steps must be carried out to find an output from the composition:

Take the reciprocal

Add 1

Take the square root

Close inspection of these steps show that these steps can be grouped so that they correspond to the functions $f(x)$ and $g(x)$:

$$\left.\text{Take the reciprocal}\right\} \quad g(x) = \frac{1}{x}$$

$$\left.\begin{array}{l}\text{Add 1}\\ \text{Take the square root}\end{array}\right\} \quad f(x) = \sqrt{x+1}$$

By listing the steps in the composition, we can group the steps to come up with two functions whose composition yields $\sqrt{\frac{1}{x}+1}$.

The decomposition is not unique. By grouping the steps in the function differently, we can come up with different functions $f(x)$ and $g(x)$ whose composition yields $\sqrt{\frac{1}{x}+1}$.

$$\left.\begin{array}{l}\text{Take the reciprocal}\\ \text{Add 1}\end{array}\right\} \quad g(x) = \frac{1}{x}+1$$

$$\left.\text{Take the square root}\right\} \quad f(x) = \sqrt{x}$$

The innermost function on the composition is always the function corresponding to the first grouping and the outermost function corresponds to the second grouping. We can check this decomposition by carrying out the composition:

$$\begin{aligned} f(g(x)) &= f\left(\frac{1}{x}+1\right) \\ &= \sqrt{\frac{1}{x}+1} \end{aligned}$$

Example 2 Write the Function as a Composition

For each part below, write given function $h(x)$ as a composition of two functions $f(x)$ and $g(x)$ so that $f(g(x)) = h(x)$.

a. $h(x) = e^{-0.015x}$

Solution Any value substituted into the function must go through a series of operations to obtain an output. If a value is substituted for x, we must first multiply the value by -0.015 and then complete e raised to the resulting value. These two operations form the two functions in the composition. The first operation corresponds to $g(x)$ and the second function corresponds to $f(x)$.

$$\text{Multiply by -0.015}\} \qquad g(x) = -0.015x$$

$$\text{Raise } e \text{ to a power}\} \qquad f(x) = e^x$$

We can carry out the composition to check these functions,

$$f\left(g(x)\right) = f\left(-0.015x\right) \qquad \text{Replace } g(x) \text{ with } -0.015x$$

$$= e^{-0.015x} \qquad \text{Substitute } -0.015x \text{ into } f(x)$$

$$= h(x)$$

The composition yields $h(x)$ so $f(x) = e^x$ and $g(x) = -0.015x$ are appropriate functions.

b. $h(x) = \dfrac{1}{x^2 + 1}$

Solution If a value is substituted into this function, several operations are carried out to yield an output.

Square the value

Add 1 to the value

Take the reciprocal

To find the two functions we'll compose to give $h(x)$, we need to group these three operations into two groups:

$$\left.\begin{array}{l}\text{Square the value}\\ \text{Add 1 to the value}\end{array}\right\} \quad g(x)$$

$$\text{Take the reciprocal}\} \quad f(x)$$

The function $g(x)$ is listed first since it is applied first when performing the composition $f\left(g(x)\right)$. Now let's add the formulas for each function:

$$\left.\begin{array}{l}\text{Square the value}\\ \text{Add 1 to the value}\end{array}\right\} \quad g(x) = x^2 + 1$$

$$\text{Take the reciprocal}\} \quad f(x) = \frac{1}{x}$$

Check these functions by carrying out the composition,

$$\begin{aligned} f\left(g(x)\right) &= f\left(x^2+1\right) && \text{Replace } g(x) \text{ with its formula } x^2+1 \\ &= \frac{1}{x^2+1} && \text{Substitute } x^2+1 \text{ into } f(x) \\ &= h(x) \end{aligned}$$

Since the composition yields $h(x)$, the functions $f(x) = \frac{1}{x}$ and $g(x) = x^2 + 1$ are appropriate.

Since $h(x)$ consists of three operations, we could also group the operations with the functions as

$$\text{Square the value} \} \quad g(x) = x^2$$

$$\left.\begin{array}{l}\text{Add 1 to the value}\\ \text{Take the reciprocal}\end{array}\right\} \quad f(x) = \frac{1}{x+1}$$

In the case of these two functions, the composition yields

$$f\left(g(x)\right) = f\left(x^2\right) \qquad \text{Replace } g(x) \text{ with its formula } x^2$$

$$= \frac{1}{x^2+1} \qquad \text{Substitute } x^2 \text{ into } f(x)$$

$$= h(x)$$

The composition $f\left(g(x)\right)$ with $f(x) = \frac{1}{x+1}$ and $g(x) = x^2$ is also appropriate. There are many possible functions we could pick for the two functions and the two possibilities worked out here are probably the most obvious.

c. $h(x) = \ln(x^2 - 2x + 1)$

Solution Start by writing out the operations involved in computing a function value for $h(x)$:

Square the original value

Subtract 2 times the original value

Add 1 to value

Take the natural logarithm of value

These four operations can be corresponded to the functions as follows:

$$\left.\begin{array}{l}\text{Square the original value}\\ \text{Subtract 2 times the original value}\\ \text{Add 1 to value}\end{array}\right\} \quad g(x) = x^2 - 2x + 1$$

$$\text{Take the natural logarithm of value}\} \quad f(x) = \ln(x)$$

We can check these functions by carrying out the composition,

$$f\left(g(x)\right) = f\left(x^2 - 2x + 1\right) \qquad \text{Replace } g(x) \text{ with its formula } x^2 - 2x + 1$$

$$= \ln\left(x^2 - 2x + 1\right) \qquad \text{Substitute } x^2 - 2x + 1 \text{ into } f(x)$$

$$= h(x)$$

The functions $f(x) = \ln(x)$ and $g(x) = x^2 - 2x + 1$ yield the composition $h(x) = f\left(g(x)\right)$.

Another possible composition corresponds to grouping the operations differently:

$$\left.\begin{array}{l}\text{Square the original value} \\ \text{Subtract 2 times the original value}\end{array}\right\} \quad g(x) = x^2 - 2x$$

$$\left.\begin{array}{l}\text{Add 1 to value} \\ \text{Take the natural logarithm of value}\end{array}\right\} \quad f(x) = \ln(x+1)$$

This choice of functions also yields $h(x)$,

$$f\left(g(x)\right) = f\left(x^2 - 2x\right) \qquad \text{Replace } g(x) \text{ with its formula } x^2 - 2x$$

$$= \ln\left(x^2 - 2x + 1\right) \qquad \text{Substitute } x^2 - 2x \text{ into } f(x)$$

$$= h(x)$$

Since the composition yields $h(x)$, the functions $f(x) = \ln(x+1)$ and $g(x) = x^2 - 2x$ are appropriate. Grouping the operations differently would lead to other appropriate functions.