Question 3: How do you apply the chain rule to take a derivative?

Functions that can be written as a composition $f\left(g(x)\right)$ are differentiated with the Chain Rule for Derivatives.

> **The Chain Rule for Derivatives**
>
> If $f(x)$ and $g(x)$ are differentiable functions, then
>
> $$\frac{d}{dx}\left[f\left(g(x)\right)\right] = f'\left(g(x)\right)g'\left(x\right)$$

In the next example, we'll use the Chain Rule for Derivatives to find a derivative. To justify the Chain Rule, we'll compute the same derivative by simplifying the function and then taking the derivative. Both strategies should lead to the same derivative.

Example 3 Apply the Chain Rule

Use $h(x) = \left(2x^3 - 7x\right)^2$ to answer the questions posed in each part.

a. Use the Chain Rule for Derivatives to find $h'(x)$.

Solution To use the Chain Rule, we need to write $h(x)$ as a composition $f\left(g(x)\right)$. If we let $f(x) = x^2$ and $g(x) = 2x^3 - 7x$, then we can compute the derivatives as $f'(x) = 2x$ and $g'(x) = 6x^2 - 7$. Now apply the Chain Rule to find the derivative $h'(x)$:

$$h'(x) = \underbrace{2\left(2x^3 - 7x\right)}_{f'(g(x))}\underbrace{\left(6x^2 - 7\right)}_{g'(x)}$$

We can simplify the function by multiplying the factors to get a polynomial:

$$h'(x) = 2\left(2x^3 - 7x\right)\left(6x^2 - 7\right)$$ Multiply the factors in parentheses

$$= 2\left(12x^5 - 14x^3 - 42x^3 + 49x\right)$$ Simplify each of the terms

$$= 24x^5 - 112x^3 + 98x$$ Combine like terms and multiply by 2 to remove the parentheses

b. Write $h(x) = \left(2x^3 - 7x\right)^2$ as a product, carry out the multiplication, and then find the derivative of the resulting polynomial.

Solution Since $\left(2x^3 - 7x\right)^2 = \left(2x^3 - 7x\right)\left(2x^3 - 7x\right)$, write the function as this product and carry out the multiplication:

$$h(x) = \left(2x^3 - 7x\right)\left(2x^3 - 7x\right)$$

$$= 2x^3 \cdot 2x^3 - 2x^3 \cdot 7x - 2x^3 \cdot 7x + 7x \cdot 7x$$ Multiply out the terms

$$= 4x^6 - 28x^4 + 49x^2$$ Simplify by multiplying factors and combining like terms

The derivative is

$$h'(x) = 24x^5 - 112x^3 + 98x$$

This derivative matches the derivative found in part a using the Chain Rule for Derivatives. ■

Example 4 Apply the Chain Rule

Use the Chain Rule for Derivatives to find each of the derivatives below.

a. $S'(Q)$ if $S(Q) = e^{0.012Q}$

Solution Write the function as a composition $f\left(g(Q)\right)$ with

$$f(Q) = e^Q \qquad g(Q) = 0.012Q$$

Notice that the independent variable for each function matches the independent variable for $S(Q)$. The derivative of each of these functions is

$$f'(Q) = e^Q \qquad g'(Q) = 0.012$$

Now apply the chain rule to find the derivative,

$$S'(Q) = \underbrace{e^{0.012Q}}_{f'(g(Q))} \cdot \underbrace{0.012}_{g'(Q)}$$

b. $D_y\left[\dfrac{1}{2y^2 - 3y + 4}\right]$

Solution Write the function as a composition $f(g(y))$ with

$$f(y) = \frac{1}{y} \qquad g(y) = 2y^2 - 3y + 4$$

The derivative of each of these functions is

$$f'(y) = -\frac{1}{y^2} \qquad g'(y) = 4y - 3$$

where we think of the first function as $f(y) = y^{-1}$ to take the derivative with the Power Rule for Derivatives. Using the Chain Rule for Derivatives, we find the derivative is

$$D_y\left[\frac{1}{2y^2 - 3y + 4}\right] = \underbrace{-\frac{1}{\left(2y^2 - 3y + 4\right)^2}}_{f'(g(y))}\underbrace{\left(4y - 3\right)}_{g'(y)}$$