Question 4: How do you combine derivative rules to take more complicated derivatives?

Many derivatives in business and economics applications will require more than a single derivative rule to differentiate. We have already seen that combining the Power Rule for Derivatives with other rules does not pose too many difficulties. When the Product Rule, Quotient Rule or Chain Rule are combined in the same problem, special care must be taken to insure each rule is applied appropriately.

The Product Rule: $\frac{d}{dx}\left[u(x)v(x)\right] = v(x)u'(x) + u(x)v'(x)$

The Quotient Rule: $\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{v(x)u'(x) - u(x)v'(x)}{\left(v(x)\right)^2}$

The Chain Rule: $\frac{d}{dx}\left[f\left(g(x)\right)\right] = f'\left(g(x)\right)g'\left(x\right)$

Example 5 Find the Marginal Revenue

In October of 2001, Apple introduced the iPod. The iPod is a digital music player. The original iPod weighed only 6.5 ounces, yet had the capability to store and play up to 1000 CD quality songs. With a battery life of 10 hours, this device became popular very quickly. By 2010, Apple was selling over 50 million iPods annually at an average selling price around \$164.

Between 2002 and 2010, the number of iPods sold annually Q (in millions) and the average selling price P (in dollars per iPod) varied according to

$$P = 341.808\,e^{-0.015Q}$$

Follow the parts below to compute and interpret the marginal revenue when 60 million iPods are sold annually.

a. Find the total revenue as a function of the number of iPods sold annually in millions of iPods.

Solution Since we are looking for the revenue as a function of the number of iPods sold, we'll name the function $TR(Q)$. To find this function, we need to use the relationship between total revenue, price and quantity,

$$\text{Total revenue} = \text{price} \cdot \text{quantity}$$

We have already named some of these quantities so let's add them to this relationship,

$$\underbrace{\text{Total revenue}}_{TR(Q)} = \text{price} \cdot \underbrace{\text{quantity}}_{Q}$$

To finish off writing out the revenue function $TR(Q)$, we need to write the price in terms of the variable Q. Luckily, this relationship is given to us as $P = 341.808\,e^{-0.015Q}$. Let's update our relationship with this information,

$$\underbrace{\text{Total revenue}}_{TR(Q)} = \underbrace{\text{price}}_{341.808e^{-0.015Q}} \cdot \underbrace{\text{quantity}}_{Q}$$

Reordering the function yields the revenue function

$$TR(Q) = 341.808Q\,e^{-0.015Q}$$

The price is in dollars per iPod and the quantity is in millions of iPods. The revenue has units of

$$\underbrace{\frac{\text{dollars}}{\cancel{\text{iPod}}}}_{\substack{\text{units on}\\ \text{price}}} \cdot \underbrace{\text{millions of } \cancel{\text{iPods}}}_{\text{units on quantity}} = \underbrace{\text{millions of dollars}}_{\text{units on revenue}}$$

This means any output from the revenue function must be in millions of dollars.

b. Find the marginal revenue as a function of the number of iPods sold annually in millions of iPods.

Solution The marginal revenue is approximated by the derivative of the revenue function. To find the derivative of the revenue function,

$$TR(Q) = 341.808Q\, e^{-0.015Q}$$

we start by applying the Product Rule for Derivatives where

$$u = 341.808Q \quad \rightarrow \quad u' = 341.808$$

$$v = e^{-0.015Q} \quad \rightarrow \quad v' = \frac{d}{dQ}\left[e^{-0.015Q}\right]$$

We'll take the derivative of v using the chain rule once we have completed the Product Rule for Derivatives. The Product Rule results in

$$TR'(Q) = \underbrace{e^{-0.015Q} \cdot 341.808}_{vu'} + \underbrace{341.808Q \cdot \frac{d}{dQ}\left[e^{-0.015Q}\right]}_{uv'}$$

The derivative in the second term, $\frac{d}{dQ}\left[e^{-0.015Q}\right]$, is computed using the Chain Rule for Derivatives. Write the expression inside the brackets as a composition with

$$f(Q) = e^{Q} \qquad g(Q) = -0.015Q$$

The derivatives of these functions are

$$f'(Q) = e^Q \qquad g'(Q) = -0.015$$

The derivative is

$$\frac{d}{dQ}\left[e^{-0.015Q}\right] = \underbrace{e^{-0.015Q}}_{f'(g(Q))}\underbrace{(-0.015)}_{g'(Q)}$$

With this derivative the marginal revenue function is

$$TR'(Q) = e^{-0.015Q} \cdot 341.808 + 341.808Q\left(-0.015e^{-0.015Q}\right)$$

This derivative can be simplified by factoring the common multiple $341.808\,e^{-0.015Q}$ from each term:

$$TR'(Q) = 341.808\,e^{-0.015Q}(1 - 0.015Q)$$

In this form, it is easier to substitute values into the function.

c. Compute the marginal revenue when 20 million iPods are sold annually. What does this tell you about the revenue?

Solution Substitute $Q = 20$ into the derivative of the revenue function to find the marginal revenue at a level of 20 million iPods,

$$\begin{aligned} TR'(20) &= 341.808\,e^{-0.015(20)}(1 - 0.015(20)) \\ &\approx 177.25 \end{aligned}$$

The units on the marginal revenue are the units on the dependent variable divided by the units on the independent variable or

$$\begin{aligned} \frac{\text{units on the revenue}}{\text{units on the quantity}} &= \frac{\cancel{\text{millions}}\ \text{of dollars}}{\cancel{\text{millions}}\ \text{of iPods}} \\ &= \frac{\text{dollars}}{\text{iPods}} \end{aligned}$$

The value $TR'(20) \approx 177.25$ means an additional iPod sold yields 177.25 dollars of additional revenue, at an annual sales level of 20 million iPods.