Question 1: Why is area important?

Let's look at the rate of change for a major company. Coca Cola is the world's largest beverage company. It produces well know soft drinks, juices, energy drinks, and bottled water. These beverages are produced in two forms. The company produces beverage concentrates that are sold to bottling operations throughout the word. These bottlers add water and sweeteners. The finished products are packaged and sold to retailers or wholesalers. In addition, they also manufacture concentrates for fountain beverages sold at restaurants and small retailers. Coca Cola also produces finished sparkling and still beverages such as juices, energy drinks, teas, and water.

Coca Cola measures the volume of beverage products sold in terms of unit cases. A unit case is equal to 192 ounces of finished beverage. Based on data from 2007 through 2011, the rate at which the gross profit has changed is given by the table below.

Quantity Q (billions of case units)	Rate of Change of Profit (Dollars per unit case)
24	2.062
25	2.279
26	2.518
27	2.782

Knowing the rate of change of profit, we can estimate the change in profit when production is increased from 24 billion case units to 25 billion case units. At these production levels, the profits is increasing at a rate of 2.062 dollars per unit case or 2.279 dollars per case.

If we assume all of the case units from $Q = 24$ billion unit cases to $Q = 25$ billion unit cases increase the profit by 2.062 dollars per unit case, then the profit increases by

$$2.062\frac{\text{dollars}}{\cancel{\text{unit case}}}\cdot 1\text{ billion }\cancel{\text{unit cases}} = 2.062\text{ billion dollars}$$

This seems reasonable since the profit is increasing by 2.062 dollars per unit case at a production level of 24 billion unit cases.

As production increases from $Q=24$ billion unit cases to $Q=25$ billion unit cases, the rate also increases. Realistically, some of the billion case units increase the profit by more than 2.062 dollars per unit case. If we assume all of the billion unit cases from 24 to 25 increase the profit by this amount, we will have a lower estimate of the increase in profit.

An upper limit on the increase in profit is found by assuming all of the unit cases from $Q=24$ billion unit cases to $Q=25$ billion unit cases increase the profit by 2.279 dollars per unit case. This is the rate at which profit is increasing at $Q=25$ billion unit cases. In this case, the profit increases by

$$2.279\frac{\text{dollars}}{\cancel{\text{unit case}}}\cdot 1\text{ billion }\cancel{\text{unit cases}} = 2.279\text{ billion dollars}$$

when production increases from 24 to 25 billion unit cases. This is an upper estimate on the increase in profit as long as the rate increases steadily from 2.062 dollars per unit case to 2.279 dollars per unit case.

Let's apply the same reasoning to a change in production from $Q=25$ to $Q=26$ billion unit cases.

$$\begin{array}{c}\text{Lower Estimate on}\\ \text{the Change in Profit}\end{array} = 2.279\frac{\text{dollars}}{\cancel{\text{unit case}}}\cdot 1\text{ billion }\cancel{\text{unit cases}} = 2.279\text{ billion dollars}$$

$$\begin{array}{c}\text{Upper Estimate on}\\ \text{the Change in Profit}\end{array} = 2.518\frac{\text{dollars}}{\cancel{\text{unit case}}}\cdot 1\text{ billion }\cancel{\text{unit cases}} = 2.518\text{ billion dollars}$$

The profit increases between 2.279 and 2.518 billion dollars as production increases from 25 to 26 billion unit cases.

Example 1 Estimate the Change in Profit

Use the table to find a lower and upper estimate of the change in profit when production is increased from 26 to 27 billion unit cases.

Quantity Q (billions of case units)	Rate of Change of Profit (Dollars per unit case)
24	2.062
25	2.279
26	2.518
27	2.782

Solution To find a lower estimate of the change in profit, assume all of the billion of unit cases from 26 to 27 billion cases increase the profit by 2.518 dollars per unit case. In this case, the profit increases by

$$2.518 \frac{\text{dollars}}{\cancel{\text{unit case}}} \cdot 1 \text{ billion } \cancel{\text{unit cases}} = 2.518 \text{ billion dollars}$$

Assume all of the unit cases from 26 to 27 billion unit cases increases the profit by 2.782 dollars per unit case. In this case, the profit increases by

$$2.782 \frac{\text{dollars}}{\cancel{\text{unit case}}} \cdot 1 \text{ billion } \cancel{\text{unit cases}} = 2.782 \text{ billion dollars}$$

In these two cases, we assume all of the unit cases increase the profit by either 2.518 dollars per unit case or 2.782 dollars per unit case. Realistically, the rate probably increases steadily from 2.518 to 2.782 dollars per unit case and the actual change is between 2.518 billion dollars and 2.782 billion dollars.

Let's summarize the estimates we have calculated.

Change in Production (billions of unit cases)	Lower Estimate of the Change in Profit (billions of dollars)	Upper Estimate of the Change in Profit (billions of dollars)
$Q = 24$ to $Q = 25$	2.062	2.279
$Q = 25$ to $Q = 26$	2.279	2.518
$Q = 26$ to $Q = 27$	2.518	2.782

If we add the lower estimates in the second column, we get a sum of changes in profit from $Q = 24$ to $Q = 27$,

$$\text{Lower Estimate} = 2.062 + 2.279 + 2.518 = 6.859$$

Similarly, the sum of the change in the third column gives an upper estimate of the change in profit when production is increased from $Q = 24$ to $Q = 27$,

$$\text{Upper Estimate} = 2.279 + 2.518 + 2.782 = 7.579$$

As long as the rate of change increases steadily, the actual change in profit is between 6.879 billion dollars and 7.579 billion dollars.

In calculating estimates, we must make sure that the lower estimates utilize the lower rate. Similarly, the upper estimate utilizes the higher rate. For the Coca-Cola estimates, the rate is increasing so the lower rate is always at the lower production level and the higher rate is at the higher production level. If the rate is decreasing, the lower rate will occur at the higher production level and the higher rate is at the lower production level. This is the case in the next example.

Example 2 Estimate the Change in Revenue

The rate of change of revenue at a small electric car company for several different production levels is given in the table below.

Quantity Q (electric cars)	Rate of Change of Revenue (thousands of dollars per electric car)
90	49.417
100	49.192
110	48.990

a. Find lower and upper estimates of the change in revenue when production in increased from 90 to 100 electric cars.

Solution When production increases from 90 to 100 electric vehicles, the rate decreases from 49.417 thousand dollars per electric vehicle to 48.990 thousand dollars per electric vehicle. The lower estimate for the change in revenue comes from assuming that each vehicle from 90 to 100 increases revenue by 49.192 thousand dollars per electric vehicle. The change in revenue is

$$\begin{aligned}\text{Lower Estimate} &= 49.192\,\frac{\text{thousand dollars}}{\cancel{\text{electric vehicle}}}\cdot 10\ \cancel{\text{electric vehicles}}\\ &= 491.92\text{ thousand dollars}\end{aligned}$$

If we asume all electric vehicles from 90 to 100 increase revenue by 49.417 thousand dollars per electric vehicle, the change in revenue is

$$\begin{aligned}\text{Upper Estimate} &= 49.417\,\frac{\text{thousand dollars}}{\cancel{\text{electric vehicle}}}\cdot 10\ \cancel{\text{electric vehicles}}\\ &= 494.17\text{ thousand dollars}\end{aligned}$$

Instead of being constant over 90 to 100 electric vehicles,he rate probably decreases steadily from 49.417 to 49.192. The actual change in revenue is probably between $491,920 and $494,170.

b. Find lower and upper estimates of the change in revenue when production in increased from 90 to 110 electric cars.

Solution In part a, we calculated the change in revenue for an increase in production from 90 to 100 electric vehicles. We can also calculate the change in revenue for an increase from 100 to 110 electric vehicles using a similar strategy.

$$\begin{aligned}\text{Lower Estimate} &= 48.990 \frac{\text{thousand dollars}}{\cancel{\text{electric vehicle}}} \cdot 10\ \cancel{\text{electric vehicles}} \\ &= 489.900 \text{ thousand dollars}\end{aligned}$$

$$\begin{aligned}\text{Upper Estimate} &= 49.192 \frac{\text{thousand dollars}}{\cancel{\text{electric vehicle}}} \cdot 10\ \cancel{\text{electric vehicles}} \\ &= 491.920 \text{ thousand dollars}\end{aligned}$$

With these estimates, we have the following changes.

Change in Production (electric vehicles)	Lower Estimate on the Change in Revenue	Upper Estimate on the Change in Revenue
$Q = 90$ to $Q = 100$	491.920	494.170
$Q = 100$ to $Q = 110$	489.900	491.920

The change in revenue from 90 to 110 electric vehicles is the sum of the individual changes from 90 to 100 electric vehicles and 100 to 110 electric vehicles. To get a lower estimate, add the individual lower estimates,

$$\text{Lower Estimate} = \underbrace{491.920}_{\substack{\text{Change in Revenue}\\ \text{from 90 to 100}}} + \underbrace{489.900}_{\substack{\text{Change in Revenue}\\ \text{from 100 to 110}}} = \underbrace{981.820}_{\substack{\text{Change in Revenue}\\ \text{from 90 to 110}}}$$

To get the upper estimate, add the individual upper estimates,

$$\text{Upper Estimate} = \underbrace{494.17}_{\substack{\text{Change in Revenue}\\ \text{from 90 to 100}}} + \underbrace{491.92}_{\substack{\text{Change in Revenue}\\ \text{from 100 to 110}}} = \underbrace{986.09}_{\substack{\text{Change in Revenue}\\ \text{from 90 to 110}}}$$

Since all of these estimates are in thousands of dollars, the actual change in revenue is between \$981,820 and \$986,090. ■

The terms in the sums we have calculated have an interesting graphical interpretation. Recall that the lower estimate for the change in profit when production is increased from 24 to 27 billion case units is

$$\text{Lower Estimate} = 2.062 \cdot 1 + 2.279 \cdot 1 + 2.518 \cdot 1$$

Each term in this sum is the product of a rate at some quantity and a change in production. For instance, the first term is the product of the rate at $Q = 24$, 2.062, and the change in production from $Q = 24$ to $Q = 25$. On a scatter plot of the rate as a function of quantity, we can visualize this product as the area of a rectangle.

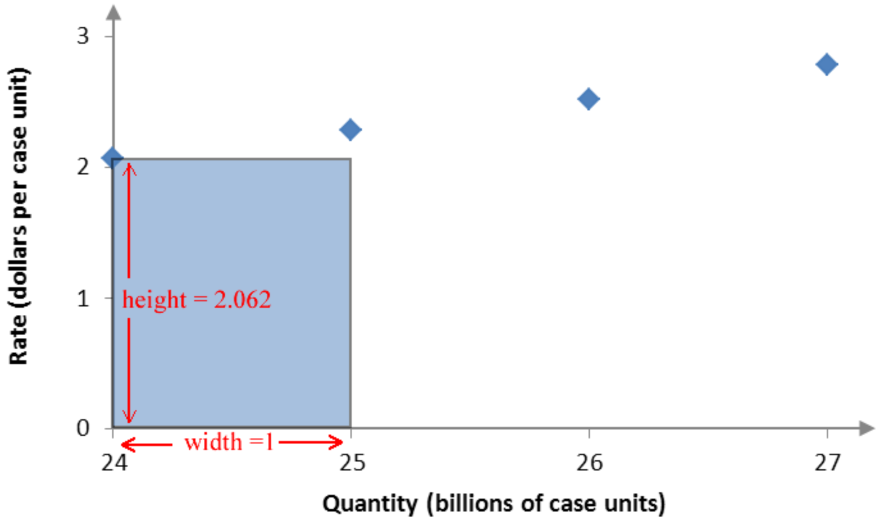

Figure 1 - The height of the rectangle corresponds to the rate at Q = 24 and the width is the change in production from Q = 24 to Q = *25*.

The area of this rectangle is $\text{height} \cdot \text{width} = 2.062 \cdot 1$.

Between the pairs of adjacent points we can place two more rectangles. In each case, the height of the rectangle comes from the height of the point on the left side of the rectangle.

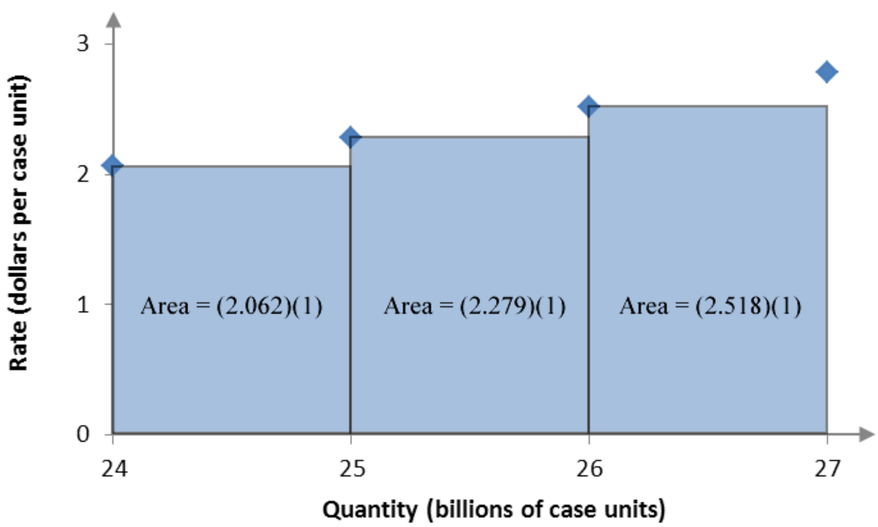

Figure 2 – In a left hand sum, the height of the rectangles are given by the height of the data point on the left hand side of the rectangle..

An estimate of this type is called a left hand sum.

The terms of the upper estimate are

$$\text{Upper Estimate} = 2.279 \cdot 1 + 2.518 \cdot 1 + 2.782 \cdot 1$$

We can also visualize each of these terms as areas of rectangles. The first term is the product of the rate at $Q = 25$, 2.279, and the change in production from $Q = 24$ and $Q = 25$. These factors are the height and width of the rectangle whose base extends from 24 to 25.

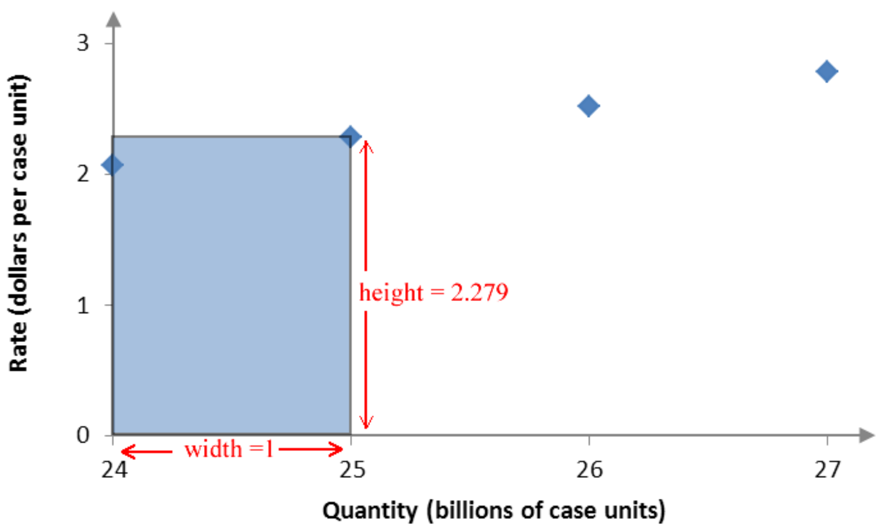

Figure 3 – The height of the rectangle is the rate at Q = 25 and the width is the change in production from Q = 24 to Q = 25.

Since this rectangle is higher than the rectangle in Figure 1, the area is greater. We can fit two more rectangles whose heights come from the point on the right hand side of the rectangle.

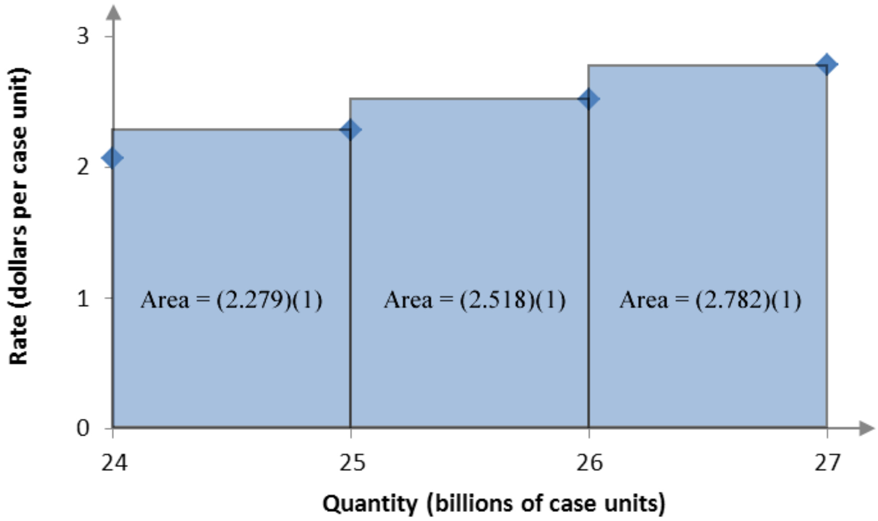

Figure 4 – In a right hand sum, the height of the rectangles are given by the height of the data point on the right hand side of the rectangle.

An estimate like this is called a right hand sum.

Left and right hand sums (and the rectangles that correspond to them) are used to calculate the change in a quantity from the rate of change of a quantity. For this example, we can symbolize the rate of change of profit as $P'(Q)$ and the change in production between two levels as ΔQ. In terms of these symbols,

$$\text{Left Hand Sum} = P'(24)\,\Delta Q + P'(25)\,\Delta Q + P'(26)\,\Delta Q$$

$$\text{Right Hand Sum} = P'(25)\,\Delta Q + P'(26)\,\Delta Q + P'(27)\,\Delta Q$$

Each sum has a similar format. The main difference is the location at which the rate is evaluated.

If we had more data points for the rate, we could evaluate the rate at different points. Suppose we extend our data table for the rate as shown below.

Quantity Q (billions of case units)	$P'(Q)$ (Dollars per unit case)
24	2.062
24.5	2.168
25	2.279
25.5	2.396
26	2.518
26.5	2.647
27	2.782

The new points in red are located midway between the points we used earlier. Instead of evaluating the rate on the left or right hand side of each rectangle, we could evaluate the rate at the midpoint of the rectangle. In this case, we call the sum a midpoint sum. The terms of this sum are

$$\begin{aligned}\text{Midpoint Sum} &= P'(24.5)\,\Delta Q + P'(25.5)\,\Delta Q + P'(26.5)\,\Delta Q \\ &= 2.168\cdot 1 + 2.396\cdot 1 + 2.647\cdot 1 \\ &= 7.211\end{aligned}$$

In this case, we would visualize the each rectangle’s height as being given by the rate at the midpoint.

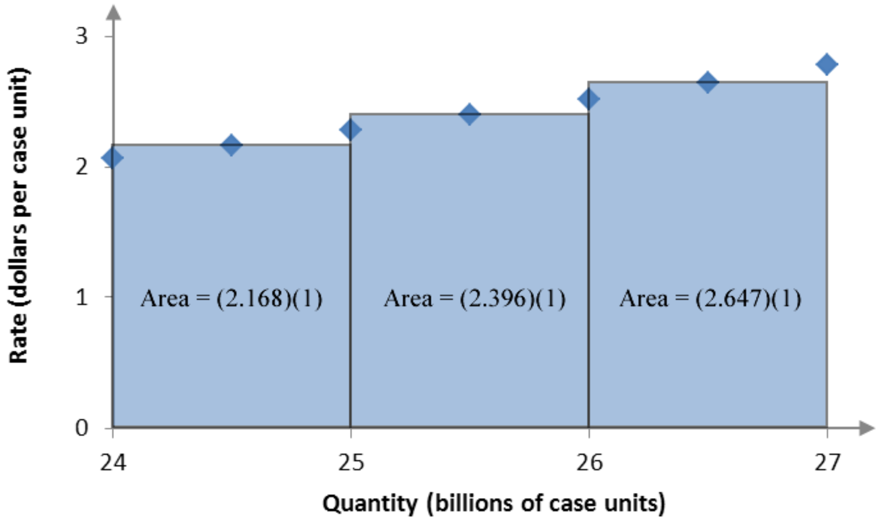

Figure 5 – In a midpoint sum, the height of each rectangle is determined by evaluating the rate at the midpoint of each rectangle.

In general, we can evaluate the rate at any point within the rectangle as long as we have the data there.

Suppose the rate is given is given by the function $f(x)$ and the data are located at values x_i, where x_i is some value in the i[th] rectangle. If each rectangle has width Δx, then area of any individual rectangle is $f(x_i)\Delta x$. The sum of the areas of n rectangles is

$$f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

Sums like this are called Riemann sums. Additionally, sums are also named by the location of the point used to determine the height of each rectangle. If the data value x_i is located on the left side of the each rectangle, the sum is called a left hand sum. If the data value x_i is located on the right side of each rectangle, the sum is called a right hand sum. If the data value x_i is located at the midpoint of the rectangle, the sum is called a midpoint sum.

In the example below, two more data points are added to the data in Example 2 so that a midpoint sum may be calculated.

Example 3 Estimate the Change in Revenue

The rate of change of revenue at a small electric car company for several different production levels is given in the table below.

Quantity Q (electric cars)	Rate of Change of Revenue (thousands of dollars per electric car)
90	49.417
95	49.301
100	49.192
105	49.089
110	48.990

Using production changes of 10, find the change in revenue using a midpoint sum when production is raised from 90 to 110 electric cars.

Solution Production changes of 10 cars corresponds to rectangles of width 10. Two rectangle can be fit from 90 to 110 as pictured below.

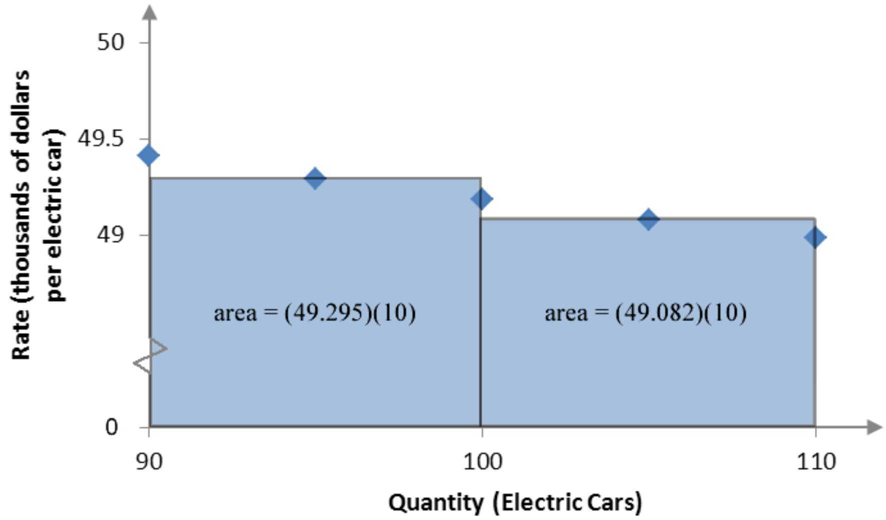

Figure 6 – In a midpoint sum, the height of the rectangle is determined by the value of the rate in the middle of the rectangle. The zig-zag on the vertical scale indicates that values have been left out on the axis.

Each rectangle's height is determined by the rate at the midpoint of the rectangle. This gives a change in revenue of

$$\begin{aligned}\text{Midpoint sum} &= 49.295 \cdot 10 + 49.082 \cdot 10 \\ &= 983.77\end{aligned}$$

The units on this value are the product of the units on the rate and quantity or

$$\frac{\text{thousands of dollars}}{\text{electric car}} \cdot \text{electric cars} = \text{thousands of dollars}$$

According to the midpoint sum, the change in revenue is approximately $983,770 when production is increased from 90 to 100 electric cars. ■