Question 1: What is a definite integral?

A definite integral is a notation used to describe the area between a function and the x axis over some interval. A definite integral uses the integral sign that was introduced earlier, but adds the endpoints of the interval. Suppose we have a non-negative function $f(x)$ with the area between $x = a$ to $x = b$ shaded.

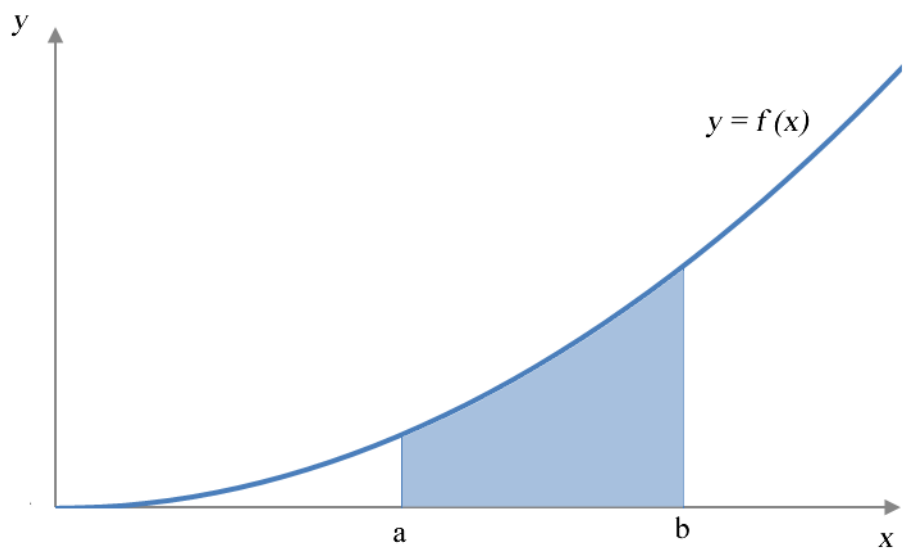


Figure 1 – The shaded area between $f(x)$ and the x axis over the interval $[a, b]$.

The definite integral for the area of this shaded region is

$$\int_a^b f(x)\,dx$$

The constants a and b are called the limits of integration. They define the interval $[a,b]$ for the shaded region. The function in the integrand, $f(x)$, forms the top boundary of the shaded region. Since this function is positive over the entire interval, the area of this region is positive.

A function that is negative has area that is negative.

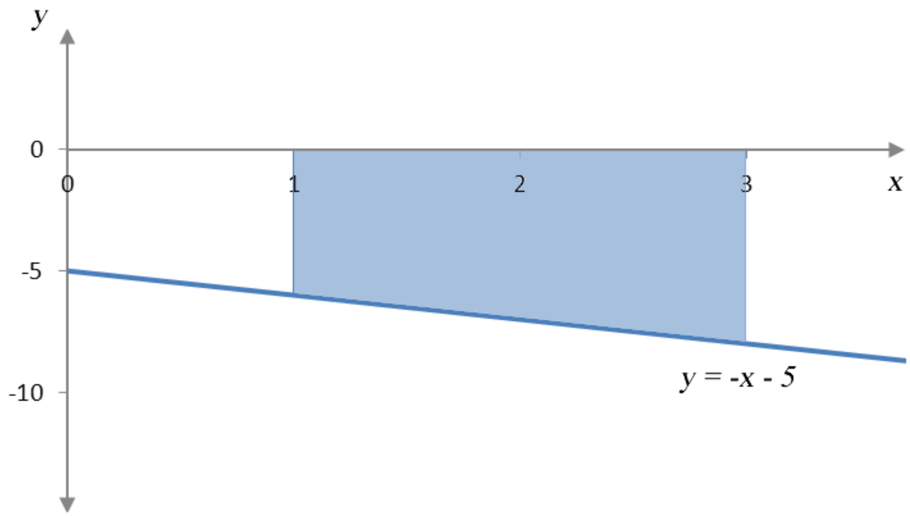


Figure 2 – The area enclosed by y = -x – 5 and the x axis, over the interval [1, 3].

The definite integral for the shaded region in Figure 2 is

$$\int_1^3 (-x-5)\,dx\,.$$

Since the function is negative over $[1, 3]$, the shaded area is negative.

Example 1 Area Between a Function and the x Axis

For each definite integral, graph the region corresponding to the area. Determine whether the definite integral is positive or negative.

a. $\int_0^2 (x^2-4)\,dx$

Solution The definite integral corresponds the region between the x axis and $y = x^2 - 4$ from $x = 0$ to $x = 2$.

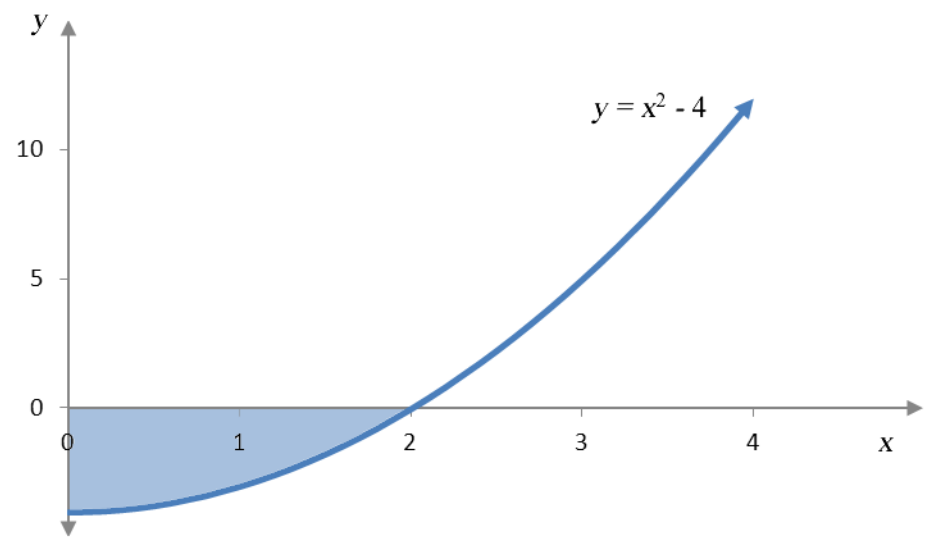


Figure 3 – The shaded region for the definite integral in part a.

Since the region is below the x axis, this area is negative.

b. $\int_{2}^{3}\left(x^2-4\right)dx$

Solution The definite integral corresponds the region between the x axis and $y=x^2-4$ from $x=2$ to $x=3$.

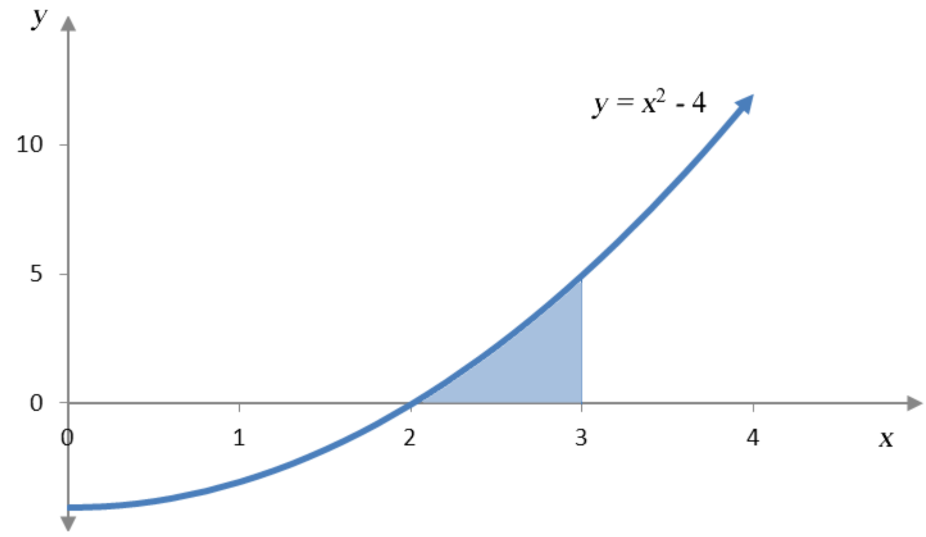
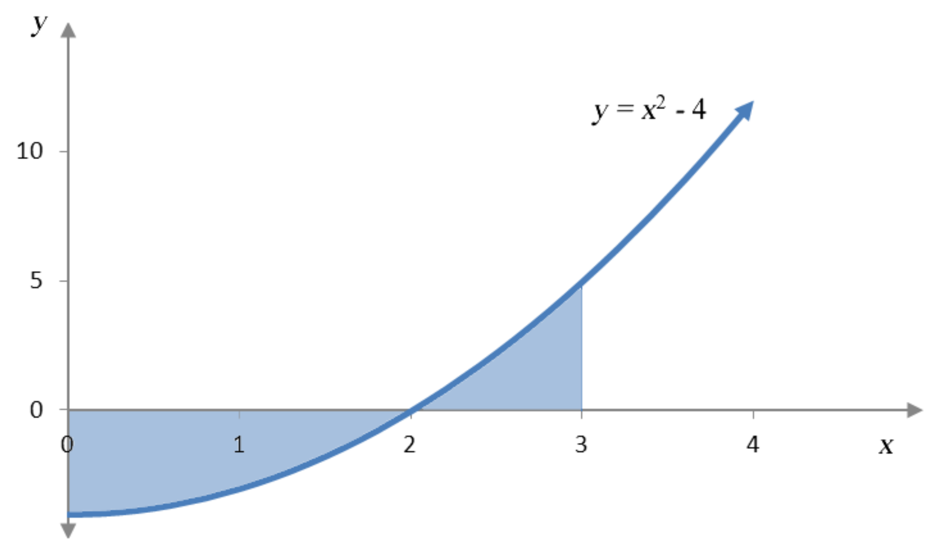


Figure 4 – The shaded region for the definite integral in part b.

Since the region is above the x axis, the area is positive.

c. $\int_0^3 \left(x^2 - 4\right) dx$

Solution This region consists of the two regions from parts a and b.



The definite integral is the sum of the area over $[0, 2]$ and $[2, 3]$. Since the negative area on $[0, 2]$ is greater than the positive area on $[2, 3]$, the definite integral is negative.

For some simple functions, the shaded region is a shape like a circle, triangle, or rectangle. For these regions, we can find the area exactly using area formulas from geometry.

Example 2 Evaluate the Definite Integral

Use area formulas to evaluate the definite integral $\int_0^5 (x+1)\,dx$.

Solution The definite integral corresponds to the area between $y = x+1$ and the x axis from $x = 0$ and $x = 5$.

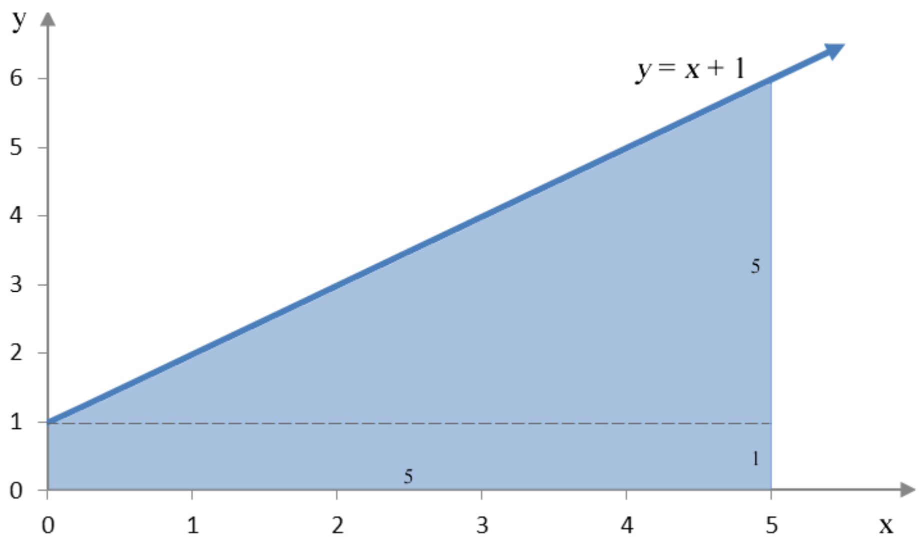


Figure 5 – The shaded region can be thought of as a rectangle capped by a triangle.

From the graph, we can see that this region consists of a rectangle and a triangle. Since we know how to find the area of each of these geometric figures, we can find the area of the region exactly.

The rectangle has a width of 5 units and length of 1 unit. Its area is

$$\begin{aligned}\text{Area of Rectangle} &= \text{length} \cdot \text{width} \\ &= (5) \quad (1) \\ &= 5\end{aligned}$$

The triangle has a base that is 5 units long. Its height is also 5 units long. The area is

$$\begin{aligned}\text{Area of Triangle} &= \frac{1}{2} \cdot \text{base} \cdot \text{height} \\ &= \frac{1}{2} \quad (5) \quad (5) \\ &= \frac{25}{2}\end{aligned}$$

The definite integral is the sum of these areas,

$$\int_0^5 (x+1)\, dx = 5 + \frac{25}{2} = 17.5$$

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For regions with more complicated shapes, we can use the strategy from 13.3 to approximate the area using rectangles.