Question 1: How is the area between two functions calculated?

In Chapter 13, we used definite integrals to find the area under a function. For a positive function $f(x)$, the definite integral

$$\int_a^b f(x)\,dx$$

corresponds to the area between $f(x)$ and the x axis from $x = a$ to $x = b$.

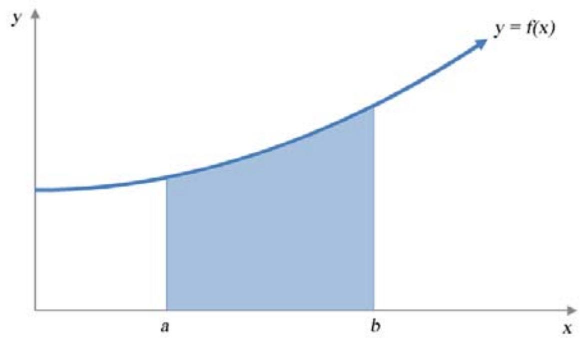


Figure 1 – The area under the function $f(x)$ and above the x axis from $x = a$ to $x = b$.

Suppose we have a smaller positive function $g(x)$ with an enclosed area

$$\int_a^b g(x)\,dx$$

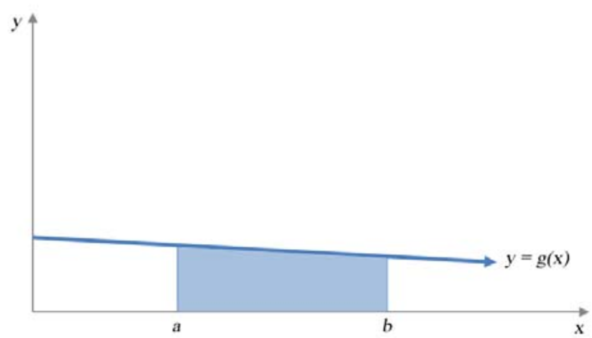


Figure 2 - The area under the function $g(x)$ and above the x axis from $x = a$ to $x = b$.

If we subtract these two areas, we get the area of the region between the functions from $x = a$ to $x = b$.

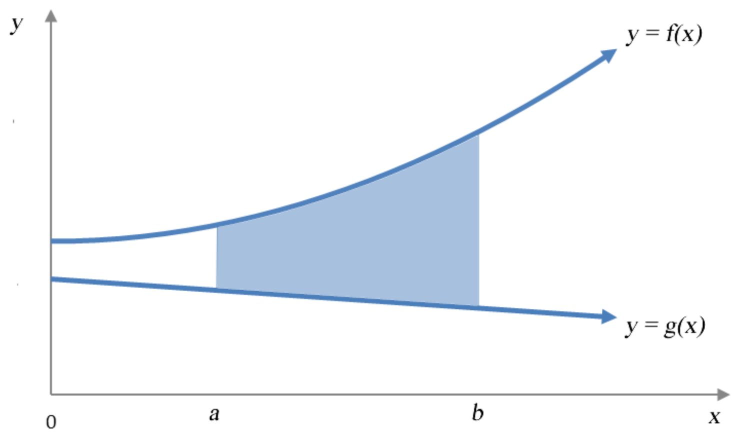


Figure 3 – The area between f(x) and g(x) from $x = a$ to $x = b$.

In terms of the individual areas, this region has an area of

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

These areas may be calculated separately and then subtracted. However, it is often more efficient to subtract the functions first and then calculate the definite integral with the difference.

The Area Between Two Curves

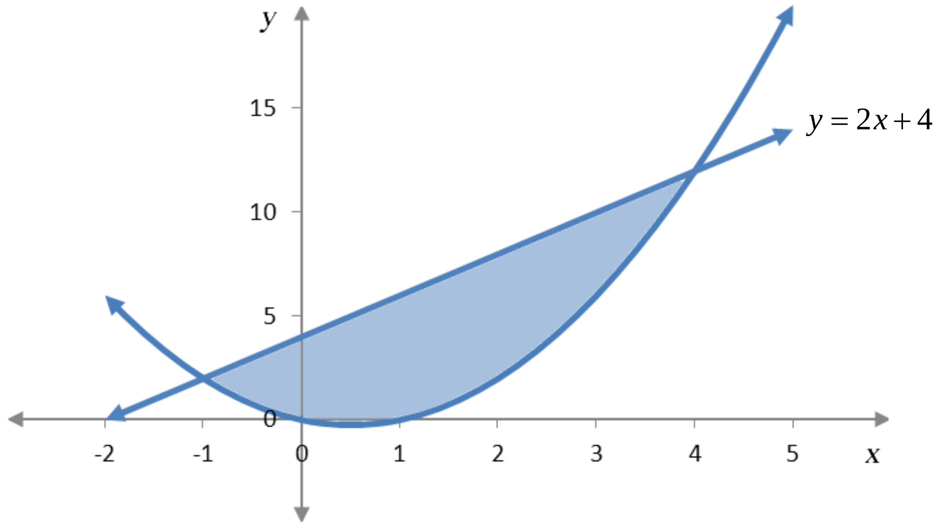
Suppose the graph of f lies above the graph of g over an interval $[a,b]$. The area of the region bounded above by $f(x)$ and below by $g(x)$ from $x=a$ to $x=b$ is

$$\int_a^b \left(f(x) - g(x) \right) dx$$

Example 1 Find the Area of the Enclosed Region

Find the area of the region below.

$y = x^2 - x$



Solution The higher function is $f(x) = 2x + 4$ and the lower function is $g(x) = x^2 - x$. From the graph, it appears that the region extends from the point of intersection on the left to the point of intersection on the right. To find the exact location of these points, set the functions equal and solve for location of these points, set the functions equal and solve for x:

$$x^2 - x = 2x + 4$$

$$x^2 - 3x - 4 = 0 \quad \text{Move all terms to one side of the equation}$$

$$(x-4)(x+1) = 0 \quad \text{Factor the trinomial}$$

$$x = 4, -1 \quad \text{Set each factor equal to 0 and solve for } x$$

These x values give us the limits of integration on the definite integral. The area of the enclosed region is

$$\int_{-1}^{4}\left[\overset{f(x)}{(2x+4)}-\overset{g(x)}{(x^2-x)}\right]dx=\int_{-1}^{4}\left[-x^2+3x+4\right]dx$$

Combine like terms in the integrand

$$=\left(-\frac{x^3}{3}+\frac{3x^2}{2}+4x\right)\Bigg|_{-1}^{4}$$

$$=\frac{56}{3}-\left(-\frac{13}{6}\right)$$

$$=\frac{125}{6}$$

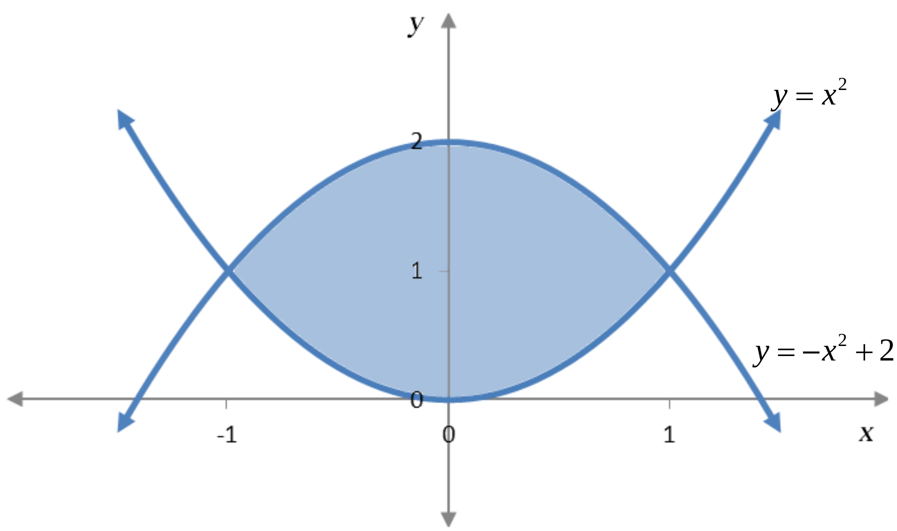
Take the antiderivative of the integrand and substitute the limits of integration

The area is $\frac{125}{6}$ or approximately 20.8333.

Example 2 Find the Area of the Enclosed Region

Find the area of the region bounded by $y=x^2$ and $y=-x^2+2$.

Solution The region is bounded above by $f(x)=-x^2+2$ and below by $g(x)=x^2$.



The graph of the region appears to extend from $x=-1$ to $x=1$. To be sure, set the functions equal and solve for x.

$$x^2 = -x^2 + 2$$

$$2x^2 - 2 = 0 \qquad \text{Move all terms to one side of the equation}$$

$$2(x-1)(x+1) = 0 \qquad \text{Factor the trinomial}$$

$$x = -1, 1 \qquad \text{Set each factor equal to 0 and solve for } x$$

Now that we know the region extends from $x = -1$ to $x = 1$, we can use a definite integral to find the enclosed area,

$$\int_{-1}^{1} \Big[\underbrace{\left(-x^2 + 2\right)}_{f(x)} - \underbrace{x^2}_{g(x)} \Big] \, dx = \int_{-1}^{1} \left[-2x^2 + 2 \right] dx \qquad \text{Combine like terms in the integrand}$$

$$= \left(\frac{-2x^3}{3} + 2x \right) \Bigg|_{-1}^{1} \qquad \text{Take the antiderivative of the integrand and substitute the limits of integration}$$

$$= \frac{4}{3} - \left(-\frac{4}{3} \right)$$

$$= \frac{8}{3}$$

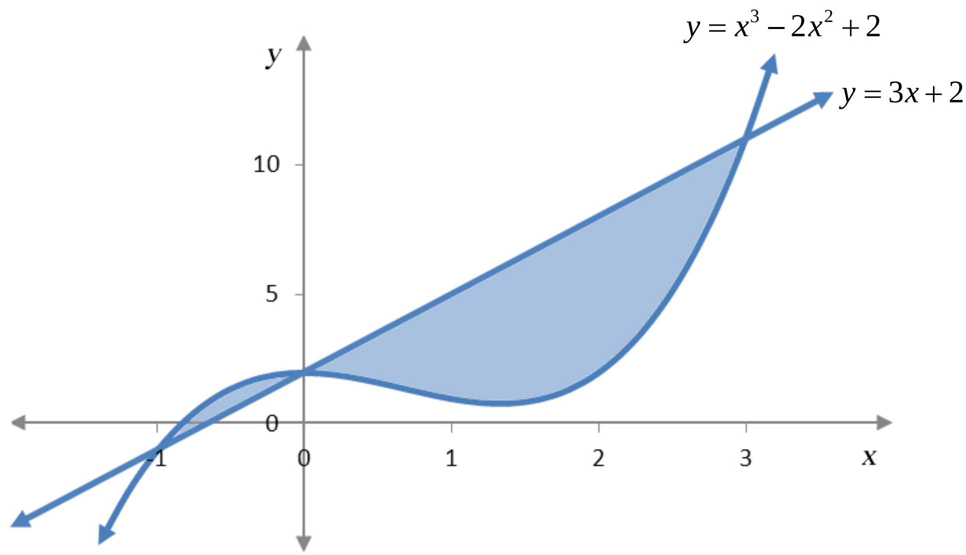
The area of the region is $\frac{8}{3}$ or approximately 2.6667.

In each of the first two examples, one function was always higher than the other function. When this is the case, the integrand is formed by subtracting the lower function from the higher function. If the functions cross to form the enclosed region, we must break the enclosed region into pieces. The area of each of the pieces is found using a definite integral and the area of each piece is added.

Example 3 Find the Area of the Enclosed Region

Find the area of the region enclosed by $y = x^3 - 2x^2 + 2$ and $y = 3x + 2$.

Solution Graph each function to see what the enclosed region looks like.



The graphs cross several times to form the enclosed region. Set the two functions equal to find the points of intersection:

$$x^3 - 2x^2 + 2 = 3x + 2$$

$$x^3 - 2x^2 - 3x = 0$$ Move all terms to one side of the equation

$$x\left(x^2 - 2x - 3\right) = 0$$ Factor the trinomial

$$x\left(x+1\right)\left(x-3\right) = 0$$ Set each factor equal to 0 and solve for x

$$x = 0, -1, 3$$

To find the area of the region that extends from $x = -1$ to $x = 0$, compute the definite integral

higher function lower function

$$\int_{-1}^{0} \left[\left(x^3 - 2x^2 + 2\right) - \left(3x + 2\right)\right] dx = \int_{-1}^{0} \left[x^3 - 2x^2 - 3x\right] dx$$ Combine like terms in the integrand

$$= \left(\frac{x^4}{4} - \frac{2x^3}{3} - \frac{3x^2}{2}\right)\Bigg|_{-1}^{0}$$ Take the antiderivative of the integrand and substitute the limits of integration

$$= 0 - \left(-\frac{7}{12}\right)$$

$$= \frac{7}{12}$$

The area of the portion of the enclosed region from $x=0$ to $x=3$ is

$$\int_0^3 \left[(3x+2)-\left(x^3-2x^2+2\right)\right]dx = \int_0^3 \left[-x^3+2x^2+3x\right]dx$$

Combine like terms in the integrand

$$= \left(-\frac{x^4}{4}+\frac{2x^3}{3}+\frac{3x^2}{2}\right)\Bigg|_0^3$$

Take the antiderivative of the integrand and substitute the limits of integration

$$= \frac{45}{4} - 0$$

$$= \frac{45}{4}$$

To find the area of the entire enclosed region, add the areas of the smaller regions, $\frac{7}{12}+\frac{45}{4}=\frac{71}{6}$, or approximately 11.8333.