Question 2: What are consumers' and producers' surplus?

An interesting application of the area between curves is consumers' and producers' surplus. To understand these concepts, we need to reacquaint ourselves with demand and supply functions. We'll do this through specific functions.

A demand function $P = D(Q)$ relates the quantity of some product Q to the price P that consumers are willing to pay for it. Suppose we have the demand function for milk in some region,

$$D(Q) = -0.05Q + 7.75 \text{ dollars per gallon}$$

where the quantity Q is in thousands of gallons. Since this function has a negative slope, as the quantity increases the price must decrease. For the consumer, when the price is cheaper they are willing to purchase more milk.

The supply function for milk is

$$S(Q) = \frac{3}{95}Q \text{ dollars per gallon}$$

This is an increasing function since the producers will supply more like as the price increases.

The demand and supply functions intersect at the equilibrium point (Q_e, P_e). At a price P_e, the consumer demands Q_e units and the producer is willing to supply Q_e units. The subscript e refers to equilibrium on each letter. We can find the values algebraically by setting the demand and supply function equal. We may also estimate the values by examining a graph.

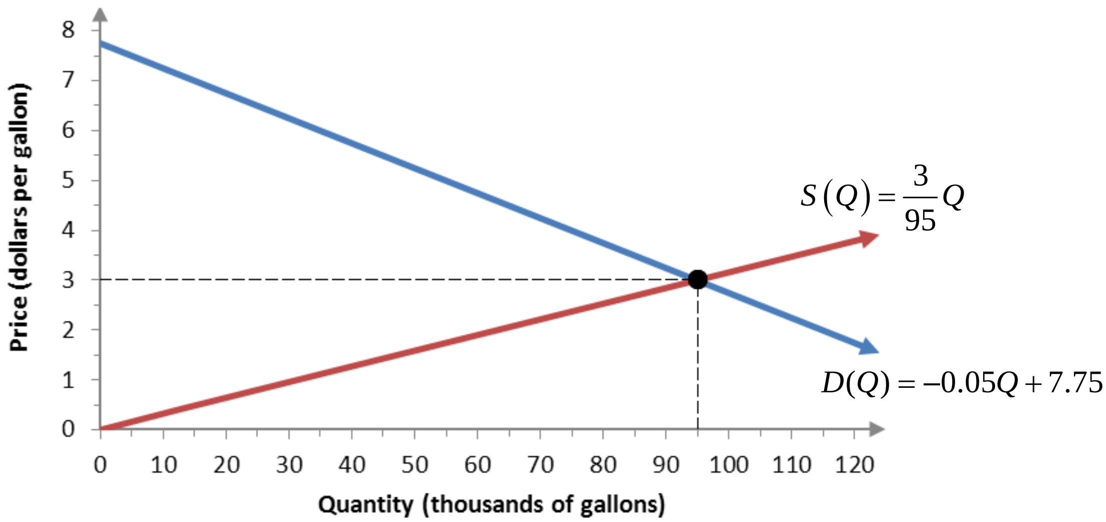

Figure 4 – The demand and supply function for milk in some market.

For the demand and supply functions for milk, the market is in equilibrium at a price of \$3 per gallon. At this price, the consumers are willing to purchase 95 thousand gallons of milk and the producers are willing to sell 95 thousand gallons. This would result in revenue for the producers of

$$\text{Revenue} = \left(3 \frac{\text{dollars}}{\cancel{\text{gallon}}}\right)\left(95 \text{ thousand } \cancel{\text{gallons}}\right) = 285 \text{ thousand dollars}$$

This is the area of the rectangle outlined by the dashed lines and axes. On this graph, area describes revenue. We could also describe this area using a definite integral,

$$\int_0^{Q_e} P_e \, dQ = \int_0^{95} 3 \, dQ$$

Set P_e equal to 3

$$= 3Q \Big|_0^{95}$$

Apply the Fundamental Theorem of Calculus to evaluate the definite integral

$$= 3 \cdot 95 - 3 \cdot 0$$

$$= 285$$

This amount assumes each gallon is sold at a constant price of \$3 per gallon.

For lower quantities the consumers would be willing to pay more than \$3 per gallon since the demand function $D(Q)$ is higher than \$3 dollars per gallon. If we compute the area under the demand function, we find the total amount of money consumers would be willing to pay for milk. The definite integral for this area is

$$\int_0^{Q_e} D(Q)\,dQ = \int_0^{95} \left(-0.05Q + 7.75\right) dQ$$

Substitute $-0.05Q + 7.75$ for $D(Q)$

$$= \left(-0.05\frac{Q^2}{2} + 7.75Q\right)\Bigg|_0^{95}$$

Apply the Fundamental Theorem of Calculus to evaluate the definite integral

$$= 510.625 - 0$$

$$= 510.625$$

The consumers would be willing to spend 510.625 thousand dollars on milk.

Although consumer's would be willing to pay \$510,625 for milk, they only pay \$285,000 by paying the equilibrium price. The amount saved by paying the equilibrium price is called the consumers' surplus

$$\text{Consumers' Surplus} = 510.625 - 285 = 225.625 \text{ thousand dollars}$$

This amount corresponds to the area between the demand function $D(Q)$ and the constant P_e from $Q = 0$ to $Q = Q_e$.

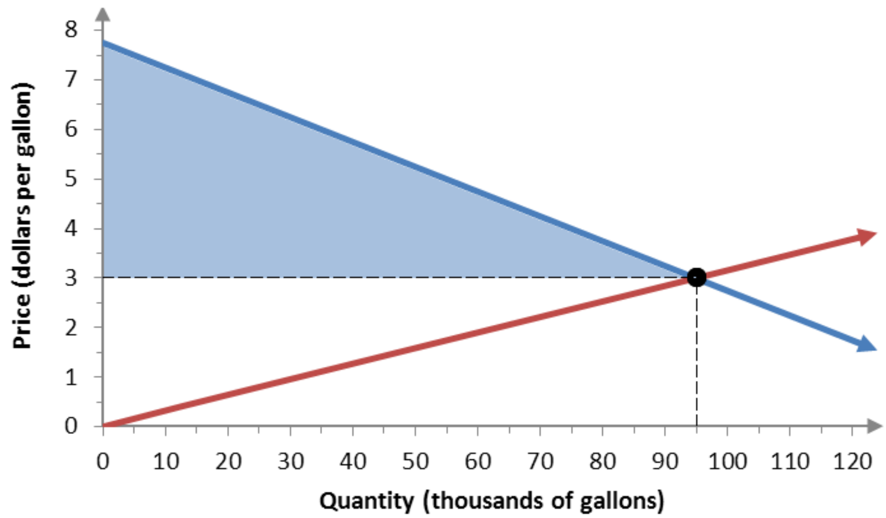

Figure 5 – The consumers' surplus is the area between the demand function and the equilibrium price from $Q = 0$ to $Q = Q_e$.

To help illustrate the meaning of this area, we have calculate the area below the demand function, below the equilibrium price, and subtracted the results. We can also find the consumers' surplus by subtracting the functions first and then computing a definite integral.

Consumers' Surplus

$$\text{Consumers' Surplus} = \int_0^{Q_e} \left(D(Q) - P_e \right) dQ$$

Example 4   Compute the Consumers' Surplus

Use the formula above to calculate the consumers' surplus for the milk demand function $D(Q) = -0.05Q + 7.75$ dollars per gallon where Q is the quantity of milk in thousands of gallons. Assume an equilibrium quantity of 95 thousand and an equilibrium price of $3 per gallon.

Solution Substitute the demand function and equilibrium point into the formula,

$$\text{Consumers' Surplus} = \int_0^{95} \left(-0.05Q + 7.75 - 3\right) dQ$$

$$= \int_0^{95} \left(-0.05Q + 4.75\right) dQ$$ Simplify the integrand

$$= \left(-0.05\frac{Q^2}{2} + 4.75Q\right)\Bigg|_0^{95}$$ Apply the Fundamental Theorem of Calculus by finding the antiderivative

$$= 225.625 - 0$$

$$= 225.625$$

The supply function describes the prices at which a producer would be willing to supply a product. The area under the supply function corresponds to the amount of revenue a producer would be willing to accept for a product. The area from $Q = 0$ to $Q = 95$ is

$$\int_0^{95} \frac{3}{95} Q \, dQ = \frac{3}{95} \cdot \frac{Q^2}{2}\Bigg|_0^{95}$$

$$= 142.5 - 0$$

$$= 142.5$$

This means that the producers would be willing to accept 142.5 thousand dollars for 95 thousand gallons of milk. However, if consumers pay the equilibrium price for all of this milk they will pay a total of 285 thousand dollars. The excess revenue the producers receive is called the producers' surplus. The producers' surplus,

$$\text{Producers' Surplus} = 285 - 142.50 = 142.50 \text{ thousand dollars}$$

corresponds the area between the constant P_e and the supply function $S(Q)$.

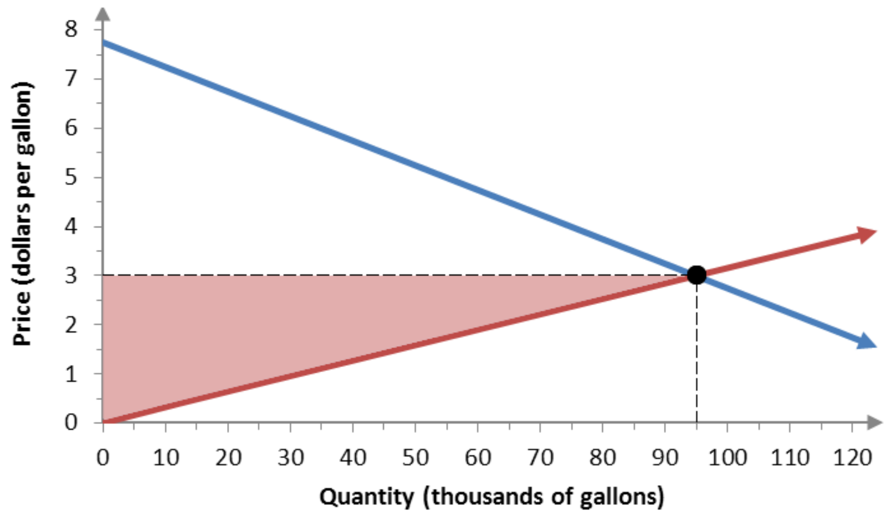

Figure 6 - The producers; surplus is the area between the equilibrium price and the supply function from $Q = 0$ to $Q = Q_e$.

Producers' Surplus

$$\text{Producers' Surplus} = \int_0^{Q_e} \left(P_e - S(Q)\right) dQ$$

Example 5 Compute the Producers' Surplus

Use the formula above to calculate the producers' surplus for the milk supply function $S(Q) = \frac{3}{95}Q$ dollars per gallon where Q is the quantity of milk in thousands of gallons. Assume an equilibrium quantity of 95 thousand and an equilibrium price of \$3 per gallon.

Solution Substitute the supply function and equilibrium point into the formula,

$$\text{Producers' Surplus} = \int_0^{95} \left(3 - \frac{3}{95}Q\right) dQ$$

$$= \left(3Q - \frac{3}{95}\frac{Q^2}{2}\right)\Bigg|_0^{95}$$

Apply the Fundamental Theorem of Calculus by finding the antiderivative

$$= 142.5 - 0$$

$$= 142.5$$