Section 2.3 Permutations and Combinations

Getting Started – What is a factorial?

- How do you determine the difference between a combination and permutation?
- How do you calculate the number of permutations of n objects taken r at a time using factorial notation and slot diagrams?
- How do you calculate the number of combinations of n objects taken r at a time?
- How do you decide which counting technique is appropriate?

Getting Started – What is a factorial?

When we apply the Multiplication Principle and do not allow repetition, the number of choices in each part of the product drops by 1. This leads to products like $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ or $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.

This type of product occurs so often that it is assigned its own symbol.

Factorial Notation

For any positive integer *n*,

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$$

The value of $0!$ is defined to be 1.

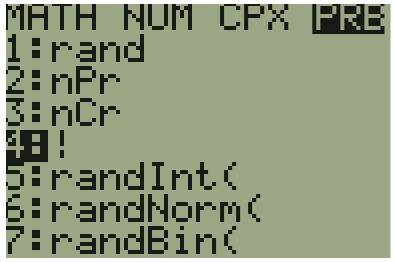
Let's look at how we might apply this to an application.

Suppose a production line requires six workers to carry out six different jobs. Each worker can only do one job at a time. Once a worker is selected for a job, the other jobs must be carried out by the remaining workers. To find the number of ways we can assign workers to jobs, calculate the product

$$\underset{\text{first job}}{\underline{6}} \cdot \underset{\text{second job}}{\underline{5}} \cdot \underset{\text{third job}}{\underline{4}} \cdot \underset{\text{fourth job}}{\underline{3}} \cdot \underset{\text{fifth job}}{\underline{2}} \cdot \underset{\text{sixth job}}{\underline{1}} = 720$$

The number of ways to make each choice drops by one in each factor since each worker can only do one job. In effect, we can't choose the same worker twice. This is often indicated by saying that we want to assign workers without repetition.

Instead of multiplying these factors out, we can utilize factorials and write it as $6!$ This may then be computed on a calculator such as a TI graphing calculator. The factorial symbol is located under the MATH button in the PRB submenu.


MATH NUM CPX PRB
1:rand
2:nPr
3:nCr
4:!
5:randInt(
6:randNorm(
7:randBin(

Notes

Guided Example 1

Evaluate each of the expressions involving factorials.

a. 8!

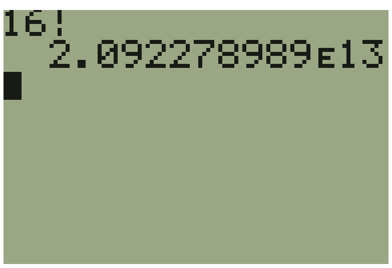
Solution Using the definition of factorial, this is equal to

$$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40320$$

b. 16!

Solution Using the definition of factorial, this is equal to

$$16 \cdot 15 \cdot 14 \cdots 3 \cdot 2 \cdot 1 \approx 2.092 \times 10^{13}$$



This is an incredibly huge number. The calculator uses scientific notation to display it. The E13 indicates $\times 10^{13}$. You would write this down by moving the decimal place 13 places to the right.

Practice

Evaluate each of the expressions involving factorials.

a. 5!

b. 20!

c. $\frac{100!}{98!}$ **Solution** Both of the numbers in the fraction are beyond most calculator's ability to calculate. However, if we use the definition of factorial we note an interesting pattern. $$\frac{100!}{98!}=\frac{100\cdot 99\cdot 98\cdot 97\cdots 3\cdot 2\cdot 1}{98\cdot 97\cdots 3\cdot 2\cdot 1}$$ Many of the factors on top match up with identical factors on the bottom. These may be reduced leaving $$\frac{100!}{98!}=100\cdot 99=9900$$	c. $\frac{82!}{80!}$

How do you determine the difference between a combination and permutation?

Key Terms

Permutations Combinations

Summary:

Permutation and Combination are two other techniques we use for counting things. They are usually used when we are selecting a subset of our original set and when we are NOT ALLOWED TO REPEAT the same item. For example, when dealing a hand of cards, you can't be dealt the same exact card twice.

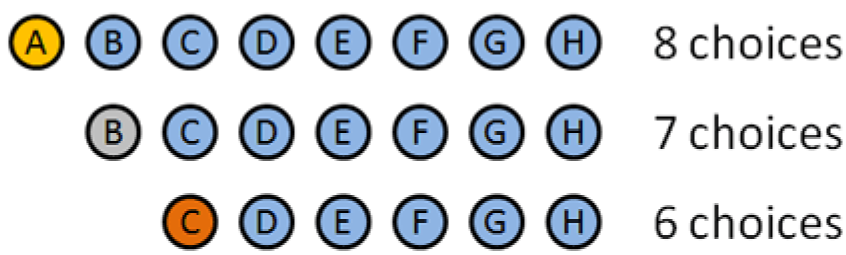
The difference between Permutations and Combinations is that for Permutations, the order of selection matters, and for Combinations it does not. This sounds easy enough, but it is usually very tricky to determine which is which. When working on problems be sure to check your answers. When you get a problem wrong, try using another technique then THINK about why you were incorrect the first time. It is OK to be wrong as long as you learn from your mistakes.

Permutation Formula	**Combination Formula**
$_nP_r = \frac{n!}{(n-r)!}$	$_nC_r = \frac{n!}{(n-r)! \cdot r!}$
where n = the total number of elements **available** r = the number of elements **selected** It can be written $_nP_r$ or P(n, r)	where n = the total number of elements **available** r = the number of elements **selected** It can be written $_nC_r$ or C(n, r)

Let's start with permutations, or **all possible ways** of doing something. We're using the fancy-pants term "permutation", so we're going to care about every last detail, including the order of each item. Let's say we have 8 people:

```
1:Alice 2:Bob 3:Charlie 4:David 5:Eve 6:Frank 7:George 8:Horatio
```

How many ways can we award a 1st, 2nd and 3rd place prize among eight contestants? (Gold / Silver / Bronze)



We're going to use permutations since the order we hand out these medals matters. Here's how it breaks down:

- Gold medal: 8 choices: A B C D E F G H (Clever how I made the names match up with letters, eh?). Let's say A wins the Gold.
- Silver medal: 7 choices: B C D E F G H. Let's say B wins the silver.
- Bronze medal: 6 choices: C D E F G H. Let's say… C wins the bronze.

We picked certain people to win, but the details don't matter: we had 8 choices at first, then 7, then 6. The total number of options was $8 \cdot 7 \cdot 6 = 336$.

Combinations are easy going. Order doesn't matter. You can mix it up and it looks the same. Let's say I'm a cheapskate and can't afford separate Gold, Silver and Bronze medals. In fact, I can only afford empty tin cans.

How many ways can I give 3 tin cans to 8 people?

Well, in this case, the order we pick people doesn't matter. If I give a can to Alice, Bob and then Charlie, it's the same as giving to Charlie, Alice and then Bob. Either way, they're equally disappointed.

So, if we have 3 tin cans to give away, there are 3*2*1 or 6 variations for every choice we pick. If we want to figure out how many combinations we have, we just **create all the permutations and divide by all the repeats**. In our case, we get 336 permutations (from above), and we divide by the 6 repeats for each permutation and get 336/6 = 56.

<u>Notes</u>

Guided Example 2	Practice
Calculate: a. $_5P_5$ **Solution** $_5P_5 = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{120}{1} = 120$ Remember that $0! = 1$ b. $_6C_4$ **Solution** $_6C_4 = \frac{6!}{2!4!} = \frac{720}{2 \cdot 24} = 15$	Calculate: a. $_6P_4$ b. $_5C_3$

Guided Example 3	Practice
a. How many permutations are there of the letters d, a, i, l, and y? **Solution** There are 5 letters in the word, so $_5P_5 = 120$. b. How many permutations are there of the letters d, a, i, l, and y if the letters are selected three at a time? **Solution** We are putting the 5 letters into 3 slots, so $_5P_3 = 60$.	a. How many permutations are there of the letters c, a, t, and s? b. How many permutations are there of the letters c, a, t, and s if the letters are selected two at a time??

Guided Example 4	Practice
Dealing three cards from a standard deck of cards, how many ways can I get three hearts? **Solution** There are 13 hearts in the deck, the order I am dealt the cards does not matter, so we use combination and calculate $_{13}C_3 = \frac{13!}{(13-3)!3!} = 286$ There are 286 different ways to get 3 hearts.	Dealing three cards from a standard deck of cards, how many ways can I get three kings?

How do you calculate the number of permutations of n objects taken r at a time using factorial notation and slot diagrams?

Key Terms:

Permutations

Summary:

If we have 24 books, how many ways can we choose four of the books? It depends on what we are going to do with the books. Are we going to display them on a bookshelf or are we going to donate them to a fundraiser?

If we are organizing them on a shelf then there is an implied order, there will be a first book, a 2^{nd} book, and so on. This indicates that we should use the Permutation formula. We have 24 total books so n = 24 and we are selected a subset of 4 books, r = 4.

$$_{24}P_4 = 225,024$$

There are 255,024 different ways of selecting four books out of a group of 24 books and lining them up.

If you thought this was a Fundamental Principle of Counting problem, you were correct. We can approach this problem from either angle. For your first book you have 24 to choose. Then for the 2nd there are only 23 left, then 22, and finally 21. Therefore, the total number of arrangements would be 24 × 23 × 22 × 21 = 255,024.

Factorial says to **multiply all whole numbers** from the chosen number down to 1.

The symbol is "!"

So we can write 24 × 23 × 22 × 21 as 24!/20!

Notes

Guided Example 5	Practice
The US Senate has 100 Senators. A Senate committee is to be formed where the committee is led by a chairman and vice chairman. How many ways is there to select the leadership of the committee? **Solution** There are two important things to note in this problem. We are choosing 2 seats, but each seat has a different responsibility. Order matters!! This is a permutation of $_{100}P_2 = 9900$. You can use a slot diagram for this problem as well: $\underline{100} \cdot \underline{99} = 9900$	The leadership of a club consists of a president, vice president and treasurer. If the club has eight members, in how many ways can the leadership be selected? Assume that the president, vice president and treasurer must all be different members.

How do you calculate the number of combinations of n objects taken r at a time?

Key Terms

Combinations

Summary

If Shaquille O'Neal runs into 20 fans, but only has 16 autographed photographs to give away, how many different ways can he give away the photographs? To get a photo you just have to be one of the 16 fans he chooses. It doesn't matter if you are the first or the 16th, either way you will get a photograph. The order does not matter. Therefore, it is a **Combination** problem.

Then n = 20 and r = 16. Thus, the number of ways Shaquille O'Neal can give away 16 photos is:

$$_{20}C_{16} = 4854$$

There are 4,845 different ways that Shaquille O'Neal can give away 16 autographed photos.

Notes

Guided Example 6	Practice
A nine-person anniversary planning committee for a community college is to be formed. Two of the members must come from 12 administrators. Three of the members must be chosen from 25 faculty members. Four of the members will be selected from 15 staff members. How many possible committees are there? **Solution** We are going to use a combination of the Fundamental Counting Principle and combinations as we have three different groups we want to combine: $_{12}C_2 \cdot {_{25}C_3} \cdot {_{15}C_4} = 66 \cdot 2300 \cdot 1365 = 207,207,000$	A Senate committee is to be formed from 52 Republican Senators and 48 Democratic Senators. If three of the members must be Republicans and two of the members must be Democrats, how many possible committees are there?

Guided Example 7	Practice
A Senate committee is to be formed from 52 Republican Senators and 48 Democratic Senators. The chairman and vice chairman must be chosen from Republicans. Two committee members must be chosen from the remaining Republican Senators and two more from the Democratic Senators. How many possible committees are there? **Solution** We are going to use a combination of the fundamental counting principle, combinations, and permutations as we have three separate events that we need to combine into a single group. First the chair and vice chair must be chosen, and because each job is unique, order matters: $_{52}P_2 = 2652$ The two committee members must be chosen from the remaining 50 senators and order does not matter as the jobs are the same: $_{50}C_2 = 1225$	A Senate committee is to be formed from 52 Republican Senators and 48 Democratic Senators. The chairman and vice chairman must be chosen from Republicans. The remaining four members of the committee are chosen from the rest of the Senate. How many possible committees are there?

Similarly, for the Democrats, two committee members from the 48 senators: $${}_{48}C_2 = 1128$$ Combining the three events using Fundamental Counting Principle: $$2652 \cdot 1225 \cdot 1128 = 3,664,533,000$$	

How do you decide which counting technique is appropriate?

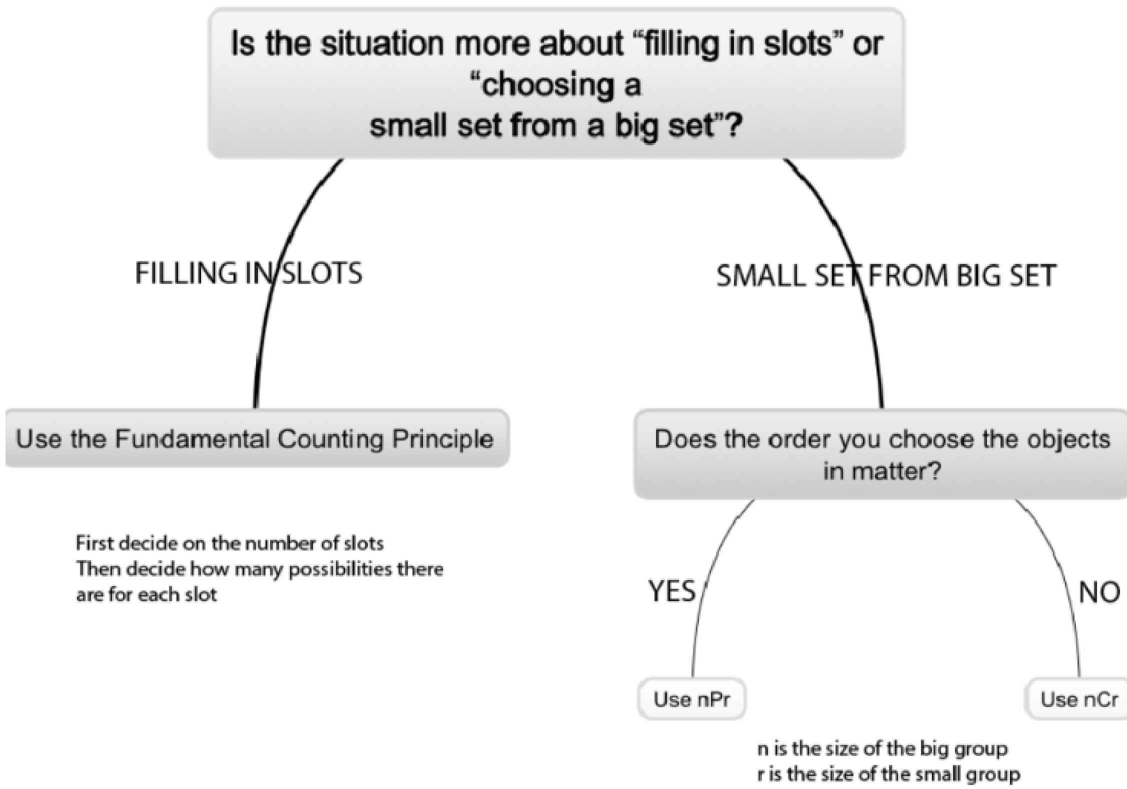
Key Terms

Fundamental Counting Principle   Combinations   Permutations

Summary

Learning when to use which counting technique takes quite a bit of practice. The flowchart below should help guide you in determining which counting technique is appropriate for which situations.

How to Determine Which Counting Technique to Use



Permutations, **Combinations**, and the **Fundamental Principle of Counting**: Remember to focus on if "order matters". Check your answers. If you get a problem wrong, it is probably because you chose the wrong formula so try another way. You need to work hard learning how to recognize when to use each different counting technique. (Note: some of the harder problems combine two counting techniques. For example, you might find a fundamental counting principle problem where you need to use a combination to calculate the number of choices for a particular slot.)

<u>Notes</u>

Guided Example 8

a. In how many different ways can a team of 11 soccer players be selected from a group of 16 athletes? (Ignore the different positions and skills)

Solution Since there are no positions assigned to the players, order does not matter. Using combinations, we get

$$_{16}C_{11} = \frac{16!}{5!(16-5)!} = 4368$$

b. In how many different ways can I arrange 12 books on my bookshelf?

Solution Since the question refers to arrangements, we implicitly assume that order makes a difference. Using permutations, we get

$$_{12}P_{12} = \frac{12!}{(12-12)!} = 479,001,600$$

c. From a group of 10 people, in how many ways can I select one person to get an A, one to get a B, and one to get a C?

Solution Since we are assigning grades, order makes a difference. Apply permutations to give

$$_{10}P_{3} = \frac{10!}{(10-3)!} = 720$$

d. From a group of 10 swimmers, how many ways can I select 3 to advance to the next round of competition?

Solution Order does not make a difference since the swimmers are not awarded prizes or ranked. Apply combinations to yield

$$_{10}C_{3} = \frac{10!}{3!(10-3)!} = 120$$

Practice

a. How many different ways can a family of four line up for a photo?

b. From a group of 10 people, in how many ways can a president, vice president, and secretary be elected?

c. From a group of 5 people, in how many ways can I select a group of three people to wash erasers?

d. In a race with 8 competitors, how many ways can 1^{st}, 2^{nd}, and 3^{rd} place be determined?

e. I'm buying a new car. I have 4 exterior colors to choose from, two different engines, three styles of interiors, and sunroof or no sunroof. How many different cars are possible? **Solution** We are filling in slots to fill with a choice of exterior color, engine, interior style, and sunroof. Using the Fundamental Counting Principle, we get $\underline{4}\cdot\underline{2}\cdot\underline{3}\cdot\underline{2}=48$	e. A menu has a choice of 5 appetizers, 6 entrees, and 4 desserts. How many meals are possible consisting of an appetizer, entrée, and dessert?