Section 2.5 Expected Value

- How do you compute the expected value to predict the long-term results of repeating an experiment?
- How do you calculate the expected value of lotteries and games of chance?

Key Terms

Random variable Expected value

Summary

Let's say Jay goes to Las Vegas and makes seven $1 bets at a roulette table. Here are the results of his gambling excursion:

–$1, –$1, –$1, $5, –1, –1, –1

If we let x equal the amount of money Jay wins, then we see that x can be –1 (when he loses the bet) or it can be 5 (it appears that when he wins, he ends up winning $5). x is called a **random variable** because it's value depends on chance (sometimes it's –1 and sometimes it's 5).

As we can see, he lost $1 six times and won $5 once. So, the average of seven bets is:

$$\frac{\sum x}{n} = \frac{-1+(-1)+(-1)+5+(-1)+(-1)+(-1)}{7} = \frac{-1}{7} \approx -\$0.14$$

If Jay were to repeat this experiment repeatedly, he would expect to lose $0.14 each time he played roulette.

> The **expected value** is the value you would "expect" to find if you could repeat the experiment an infinite number of time and take the average of the values obtained.

How do we calculate expected value of a random variable x?

1. Decide on exactly what x represents. Look for a value assigned to each outcome in the experiment.
2. Determine what the possible values of x are. These values are represented by $x_1, x_2, \cdots, x_n$.
3. Determine the probability corresponding to each x. The values are represented by $p_1, p_2, \cdots, p_n$. Note: All the probabilities should add up to one.
4. The expected value of the random variable is calculated by multiplying the value of each outcome times the corresponding probability and then adding the results:

$$E = x_1 p_1 + x_2 p_2 + \cdots + x_n p_n$$

Notes

Guided Example 1

Below are the probabilities and values associated with four outcomes of an experiment.

Outcome	Value	Probability
A	5	0.2
B	3	0.3
C	2	0.4
D	-5	0.1

Find the expected value of the experiment.

Solution We start by multiplying the probability by the value.

$$5 \cdot 0.2 = 1.0$$

$$3 \cdot 0.3 = 0.9$$

$$2 \cdot 0.4 = 0.8$$

$$-5 \cdot 0.1 = -0.5$$

The expected value is the sum of these products,

$$E = 1.0 + 0.9 + 0.8 + (-0.5) = 2.2$$

Practice

Below are the probabilities and values associated with five outcomes of an experiment.

Outcome	Value	Probability
A	1000	0.01
B	2	0.2
C	3	0.15
D	5	0.1
E	-2	0.54

Find the expected value of the experiment.

<table>
<tr><th><u>Guided Example 2</u></th><th><u>Practice</u></th></tr>
<tr><td>A biology test consists of several multiple-choice questions. Correct answers earn one point each. Questions left blank neither receive nor lose points. There are five options for each question and wrong answers are penalized $\frac{1}{4}$ point each. If a student guesses on each question, how many points can they expect on a 25-question test?

Solution When a question is attempted, there are three possible outcomes: answer correct, answer incorrect, or no answer. Each of these outcomes is associated with a point value described in the problem.

To calculate the expected value, we need to know the probability of each outcome. We start with finding the probability of guessing correctly on each individual question. Because there is one correct answer, there is a $\frac{1}{5}$ probability of guessing correctly and $\frac{4}{5}$ probability of guessing. The likelihood of not answering is not knowable. However, since there is no penalty for leaving an answer blank it does not affect the expected value. We can summarize this information in the table below.

<table><tr><th>Outcome</th><th>Value</th><th>Probability</th></tr><tr><td>Answer correct</td><td>1</td><td>$\frac{1}{5}$</td></tr><tr><td>Answer incorrect</td><td>$-\frac{1}{4}$</td><td>$\frac{4}{5}$</td></tr><tr><td>No answer</td><td>0</td><td>?</td></tr></table>
The expected value is calculated by multiplying the values times their corresponding probabilities and summing the products:
$$E = 1\cdot\frac{1}{5} + (-\frac{1}{4})(\frac{4}{5}) + 0\cdot ? = 0$$
This tells us that the average number of points you will earn by guessing for each question is zero. So, the score you would expect on a 20-question test is $20\cdot 0$ or 0.

They can expect a zero if they guess on every question.</td><td>A geology test consists of several multiple-choice questions. Correct answers earn one point each. Questions left blank neither receive nor lose points. There are four options for each question and wrong answers are penalized $\frac{1}{3}$ point each. If a student guesses on each question, how many points can they expect on a 20-question test?</td></tr>
</table>

How do you calculate the expected value of lotteries and games of chance?

Key Terms

Fair game

Summary

The expected value is the value you would "expect" to find if you could repeat the experiment an infinite number of times and take the average of the values obtained.

When we play a game with a set of outcomes, the values assigned to those outcomes may be amounts of money that you might win or lose when the game is played. The amounts may be positive if you will have a net gain or negative if you have a net loss. If the expected value of the game is 0, the game is called a **fair game**.

Games that are not fair may be made fair by modifying the bet that is wagered to play the game. For instance, suppose you pay a dollar to play a game that has an expected value of -0.25. This means you will lose 0.25 on average each time you play the game. If you adjust the bet down by 0.25 to 0.75, the game will become fair (and have an expected value of 0).

Notes

Guided Example 3

Suppose you pay $5 to play a game in which a pair of fair dice is rolled.

- If doubles (same number on each die) comes up, then you win $10.
- If an odd total comes up, then you win $6.
- For all other rolls you lose the amount you paid to play the game.

a. Find the expected value of the game.

Solution Let's describe the outcomes and values of the game. Make sure the values include the bet by making the value the "net" amount.

Outcome	Value	Probability
Doubles	$5	
Odd total	$1	
All other rolls	$-5	

For instance, even though you win $10 for getting doubles, you net $5 since you bet $5 to play the game.

To determine the probability of each outcome, consult a table of possible dice rolls and their corresponding sums.

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Of the 36 possible rolls, six of them are doubles, eighteen are odd totals, and the other twelve are something else.

We can finish our table with the corresponding probabilities.

Practice

Suppose you pay $1 to play a game in which a pair of fair dice is rolled.

- If a total of 7 or 11 comes up, then you win $5.
- If a total of 12 comes up, then you win $10.
- For all other totals you lose the amount you paid to play the game.

a. Find the expected value of the game.

Outcome	Value	Probability
Doubles	$5	$\frac{6}{36}$
Odd total	$1	$\frac{18}{36}$
All other rolls	$-5	$\frac{12}{36}$

Multiply the values times the corresponding probabilities and add the results to get the expected value,

$$E = 5 \cdot \frac{6}{36} + 1 \cdot \frac{18}{36} + (-5) \cdot \frac{12}{36} = -\frac{12}{36}$$

or approximately$-0.33. This means you will lose 0.33 dollars each time you play the game.

b. If the game is not fair, what should you pay to make the game fair?

Solution Since the expected value is not 0, the game is not fair. To make it fair, you need to pay 0.33 dollars less to play it. So, a bet of $5 – 0.33 or about $4.67 would make the game fair.

b. If the game is not fair, what should you pay to make the game fair?

Guided Example 4	Practice
One hundred tickets are sold at $2 apiece for a raffle. There is a grand prize of $50, two second place prizes of $20, and four third place prizes of $10. a. Calculate the expected value of the lottery.	One thousand tickets are sold at $5 apiece for a raffle. There is a grand prize of $2000, two prizes of $1000, and four prizes of $100. a. Calculate the expected value of the lottery.

Solution Construct a table of outcomes, values and probabilities for the lottery.

Outcome	Value	Probability
Grand Prize	$48	$\frac{1}{100}$
Second Prize	$18	$\frac{2}{100}$
Third Prize	$8	$\frac{4}{100}$
Lose	$-2	$\frac{93}{100}$

The probabilities are based on the total number of tickets available and the number of winning tickets in each outcome.

The expected value is

$$E = 48 \cdot \frac{1}{100} + 18 \cdot \frac{2}{100} + 8 \cdot \frac{4}{100} + (-2) \cdot \frac{98}{100} = -\frac{80}{100}$$

On average, you will lose 80 cents on each lottery ticket.

b. If the game is not fair, determine a price for a ticket that would make it fair.

Solution To make the ticket fair, you must lower the price by 80 cents. This would make the cost of the ticket $2 - $0.80 or $1.20.

Practice:

b. If the game is not fair, determine a price for a ticket that would make it fair.