Section 4.3 Modeling with Quadratic Equations

- What is the quadratic formula?
- How do you graph a quadratic equation?
- How can you model data with a quadratic equation?

What is the quadratic formula?

Key Terms

Quadratic Equation Quadratic Formula

Summary

Linear equations are convenient and easy to work with, but not all growth is linear. Our next type of growth is quadratic growth. We begin with an overview of basic characteristics and techniques of quadratic equations.

A quadratic equation is an equation of the form $y = ax^2 + bx + c$ **.** Notice the highest power of x is x^2, which means you can have up to two solutions to a quadratic equation. To solve quadratic equations, we use the quadratic formula:

> **Quadratic Formula**: The solution to an equation of the form $ax^2 + bx + c = 0$ (where a is not zero) is given by the formula,
>
> $$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Notice the equation must be set equal to zero before the formula is valid.

Notes

Guided Example 1	Practice
Solve the equation $x^2 - 5x + 6 = 0$. **Solution** Notice in this equation $a = 1,\ b = -5,\ c = 6$. Substituting these values into the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ we have: $x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(6)}}{2(1)}$ $= \frac{5 \pm \sqrt{25-24}}{2}$ $= \frac{5 \pm \sqrt{1}}{2}$ This gives us two results: $x = \frac{5+1}{2}$ and $x = \frac{5-1}{2}$. Carrying out the arithmetic gives $\frac{5+1}{2} = \frac{6}{2} = 3$ and $\frac{5-1}{2} = \frac{4}{2} = 2$ The solutions are $x = 3$ and $x = 2$.	Solve the equation $x^2 + 8x + 16 = 0$

Guided Example 2	Practice
Solve the equation $3x^2+10x-8$. **Solution** Notice in this equation $a=3,\ b=10,\ c=-8$ Substituting these values into the quadratic formula $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ we have: $x=\dfrac{-10\pm\sqrt{10^2-4(3)(-8)}}{2(3)}$ $=\dfrac{-10\pm\sqrt{100+96}}{6}$ $=\dfrac{-10\pm\sqrt{196}}{6}$ $=\dfrac{-10\pm14}{6}$ This gives us two results: $x=\dfrac{-10+14}{6}$ and $x=\dfrac{-10-14}{6}$ Doing the arithmetic gives the solutions $\dfrac{-10+14}{6}=\dfrac{4}{6}=\dfrac{2}{3}$ and $\dfrac{-10-14}{6}=\dfrac{-24}{6}=-4$ The solutions are $x=\dfrac{2}{3}$ and $x=-4$	Solve the equation: $6x^2-x-15=0$.

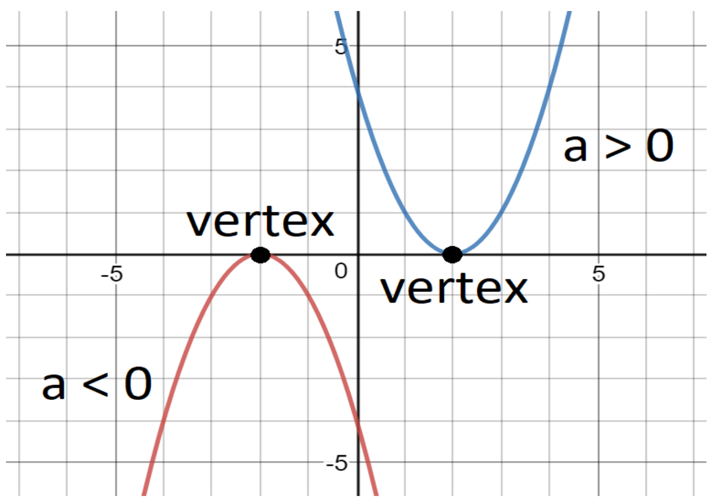
How do you graph a quadratic equation?

Key Terms

Parabola Vertex

Summary

We will investigate the graph of a quadratic equation in order to better understand the nature of quadratic growth. The graph of a quadratic equation is called a **parabola**, examples of which can be seen in the following graphic. A parabola opens up or down depending on the sign of a, and the most important point on a parabola is called the **vertex**, which is the maximum or minimum point depending on the orientation of the parabola.



The vertex can be located using the following formula:

Vertex Formula: The vertex of a parabola occurs at the x-coordinate,

$$x = \frac{-b}{2a}$$

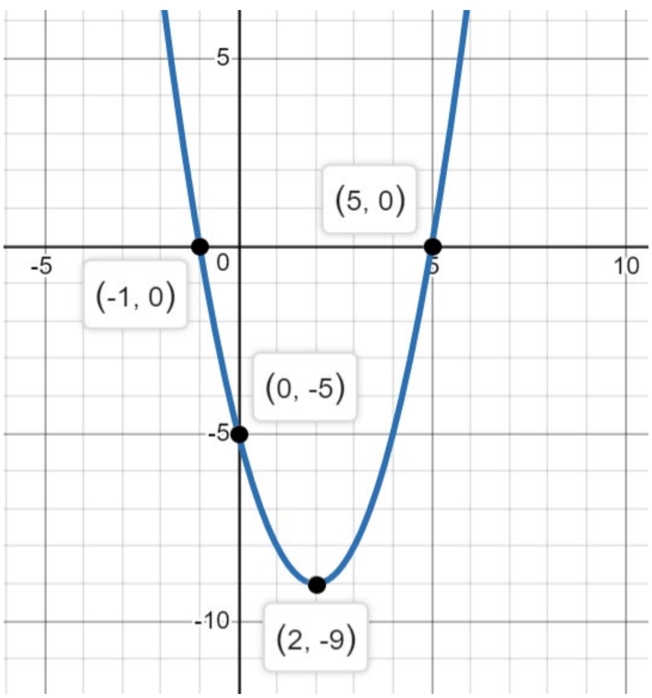
Once you calculate the x-coordinate, you can substitute that value for x in the quadratic equation to solve for y.

The last details of the graph we may be interested in are the x and y intercepts. We can solve for these as we did in section 4.1, however notice that we will need to use the quadratic formula to solve for the x-intercepts.

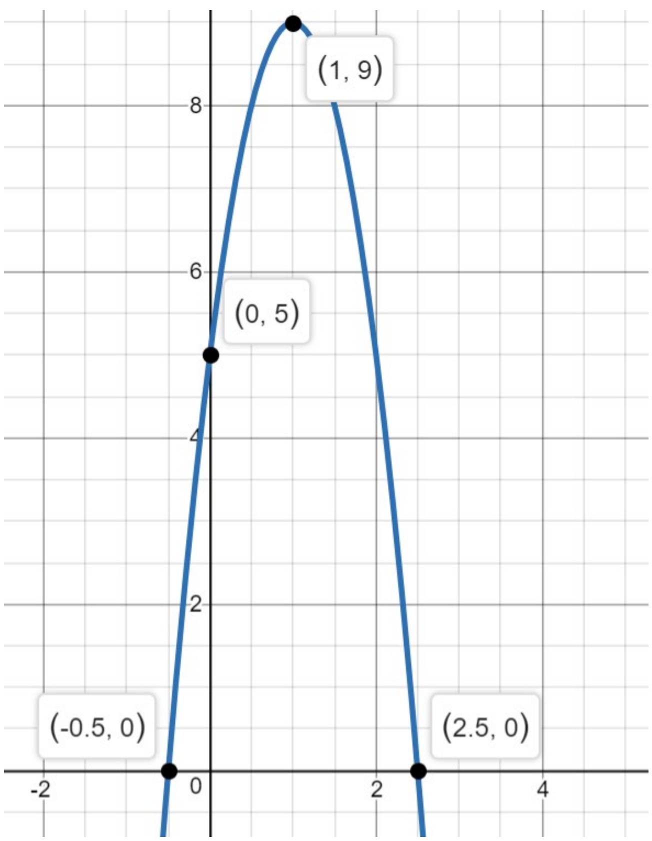
To summarize, if we want to graph a parabola, we need to determine the following characteristics of the equation:

1. What is the orientation of the parabola (opening up or down)?
2. Where is the vertex located?
3. Where are the *x*- and *y*-intercepts located?

Guided Example 3	Practice
Sketch a graph of the equation $y = x^2 - 4x - 5$ by finding the vertex and the *x*- and *y*-intercepts. Use these points to graph the parabola. **Solution** Notice in this equation that $a = 1$, $b = -4$, and $c = -5$. Since *a* is positive we know the parabola opens upwards. Next, we find *x*-coordinate of the vertex using $x = -\dfrac{b}{2a}$: $$x = \frac{-(-4)}{2(1)} = \frac{4}{2} = 2$$ Substitute this value into the equation $y = x^2 - 4x - 5$ to find the corresponding *y*-coordinate: $$y = (2)^2 - 4(2) - 5 = -9$$ This makes the vertex $(2, -9)$ Next, we find the *x*-intercepts by setting *y* to zero, $$0 = x^2 - 4x - 5$$ and solving for *x* using the quadratic formula: $$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-5)}}{2(1)}$$ Set $a = 1$, $b = -4$, and $c = -5$. $$= \frac{4 \pm \sqrt{16 + 20}}{2}$$ Simplify under the root $$= \frac{4 \pm \sqrt{36}}{2}$$	Sketch a graph of the equation $y = -x^2 + 6x - 8$ by finding the vertex and the *x*- and *y*-intercepts. Use these points to graph the parabola.

Carry out the square root to give $x = \frac{4 \pm 6}{2}$. Using the plus and minus signs separately gives two solutions: $$x = \frac{4+6}{2} = 5 \text{ and } x = \frac{4-6}{2} = -1$$ The x-intercepts are (5, 0) and (-1, 0). Lastly, we find the *y*-intercept by setting *x* to zero in $y = x^2 - 4x - 5$ and solving for *y*: $$y = (0)^2 - 4(0) - 5 = -5$$ The y-intercept is (0, -5). Now that we have calculated all of the details, we will plot the points on the graph and sketch a smooth curve to connect them: 	

Guided Example 4	Practice
Sketch a graph of the equation $y=-4x^2+8x+5$ by finding the vertex and the *x*- and *y*-intercepts. Use these points to graph the parabola **Solution** Notice in this equation $a=-4$, $b=8$, and $c=5$. Since *a* is negative we know the parabola opens downwards. Find *x*-coordinate of the vertex using $x=-\frac{b}{2a}$: $$x=-\frac{8}{2(-4)}=\frac{-8}{-8}=1$$ Substitute this value into the equation $y=-4x^2+8x+5$ to find the *y*-coordinate: $$0=-4x^2+8x+5$$ Putting these values together places the vertex at $(1,9)$. Next, we find the *x*-intercepts by setting *y* to zero, $0=-4x^2+8x+5$, and solving for *x* using the quadratic formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$: $$x=\frac{-8\pm\sqrt{8^2-4(-4)(5)}}{2(-4)}$$ $$=\frac{-8\pm\sqrt{64+80}}{-8}$$ $$=\frac{-8\pm\sqrt{144}}{-8}$$ $$=\frac{-8\pm 12}{-8}$$ Find each of the solutions: $$x=\frac{-8+12}{-8}=-\frac{1}{2} \text{ and } x=\frac{-8-12}{-8}=\frac{5}{2}$$	Sketch a graph of the equation $y=x^2-4x-12$ by finding the vertex and the *x*- and *y*-intercepts. Use these points to graph the parabola

The x-intercepts are (given as decimals to make graphing easier): $(-0.5, 0)$ and $(2.5, 0)$. Lastly, we find the *y*-intercept by setting *x* to zero in $y = -4x^2 + 8x + 5$ and solving for *y*: $$y = -4(0)^2 + 8(0) + 5 = 5$$ The y-intercept is $(0, 5)$. Now that we have calculated all of the details, we will plot the points on the graph and sketch a smooth curve to connect them:  	

How can you model data with a quadratic equation?

Key Terms

Summary

Consider a parabola that opens down ($a < 0$): It begins by increasing rapidly, but the rate of increase slows until it reaches its maximum value (vertex). After the vertex it, begins decreasing, slowly at first, but the rate of decrease continues to speed up. Any phenomena that exhibits similar behavior is a good candidate to be modeled with a quadratic equation. For example, a cannon ball shot up into the air will rapidly increase in height, then slow down and reach a maximum height, and then start to fall faster and faster back to the ground. A parabola that opens upwards ($a > 0$) exhibits similar behavior in which it decreases to a minimum value (vertex) and then starts to increase again.

When using quadratic models, often we are interested in the maximum or minimum value of the model, which would require finding the vertex using $x = -\frac{b}{2a}$. We may also want to know when the quantity we are measuring reaches a specific value. This requires the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Below we investigate some of these cases.

Notes

Guided Example 5

A cannonball is fired directly upward with an initial velocity of 500 feet per second. Its height above the ground at time *t* can be modeled with the equation $h = -16t^2 + 500t$. How high does the cannonball travel before it begins to fall back to the ground?

Solution We are asked to find the maximum height that the cannonball travels, and since our parabola opens downwards (*a* is negative) we must find the vertex. Notice in our equation $a = -16,\ b = 500,\ c = 0$. Using the vertex formula $x = -\dfrac{b}{2a}$, we have:

$$x = -\frac{500}{2(-16)} = -\frac{500}{-32} = 15.625$$

The cannonball reaches its maximum height after 15.625 seconds. To find the height at 15.625 seconds, we will substitute this time into the equation $h = -16t^2 + 500t$ and solve for *h*:

$$h = -16(15.625)^2 + 500(15.625) = 3906.25$$

The maximum height that the cannonball will reach before travelling back to the ground is 3,906.25 feet.

Practice

Assume the equation $S = -4x^2 + 16x - 2$ models the sales of a recently released video game where *x* is the number of weeks after release, and *S* is the number of sales in millions. According to this model, what are the peak sales for the game?

Guided Example 6

Suppose a company's stock is priced at $250 per share before the CEO makes an unpopular comment in an interview that results in a drop in the value of the stock. After some time passes, the stock starts to rise again. Financial analysts model the stock's behavior with the equation $y = 19x^2 - 75x + 250$, where *x* is the number of months after the interview, and *y* is the price of the stock. How many months will it take for the

Practice

A video goes viral on social media, and the number of views per day can be modeled with the equation $y = -0.125x^2 + 1.75x$, where *x* is the number of days after the video is posted, and *y* is the number of views per day in thousands. According to the model, after how many days do people stop watching the video?

stock to return to its value before the unpopular comment was made? **Solution** Notice before the interview we are told the stock was valued at \$250 per share, so we are being asked to find the number of months (x) until the stock reaches \$250 per share again. We will substitute 250 into the equation for y and solve for x. This will require the quadratic formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, but first we must set the equation equal to zero: $250=19x^2-75x+250$ Subtract 250 from each side. $0=19x^2-75x$ Apply the quadratic formula equation using $a=19$, $b=-75$, and $c=0$: $x=\frac{-(-75)\pm\sqrt{(-75)^2-4(19)(0)}}{2(19)}$ $=\frac{75\pm\sqrt{5625}}{38}$ Simplify within the square root $=\frac{75\pm 75}{38}$ Take the square root $=0, 3.9$ Work out each solution The solution $x=0$ represents the fact that before the interview (zero months after it occurred) the price of the stock was \$250. The solution we want is $x=3.9$ (rounded), which means about 3.9 months after the interview, the stock returns to \$250 per share.	