Section 4.4 Exponential Equations and Growth

- What is the difference between linear, quadratic, and exponential growth?
- How can exponential equations model growth?
- How are exponential equations solved using logarithms?

What is the difference between linear, quadratic, and exponential growth?

Key Terms

Compound Growth   Compound Interest

Summary

Exponential growth involves **compound growth**, that is, growth on top of previous growth. For example, if you invest $1,000 in an account that earns 10% interest compounded annually, in the first year you would earn 10% of $1,000, which is $100. Now you have $1,100 in the account. In the second year, you would earn 10% of $1,100, which is $110, so now you have $1,210 in the account. Notice each year you are earning interest on the original $1,000 *plus the previous interest you have already accumulated*. This is known as **compound interest**.

This same principal can be applied to population growth, and many other scenarios. Because the growth rate is applied repeatedly for each time unit, we can simplify the calculation by using exponents, which is why we use the term exponential growth.

Let us summarize our three types of growth:

- **Linear growth**: a quantity changes by the same *constant amount* for each unit of time.
- **Exponential growth**: a quantity changes by the same *percentage or factor* for each unit of time.
- **Quadratic growth**: a quantity *increases, reaches a maximum, then decreases* or *decreases, reaches a minimum, then increases*.

Notes

Guided Example 1	Practice
Determine which of the scenarios should be modeled with a linear, quadratic, or exponential equation:	Determine which of the scenarios should be modeled with a linear, quadratic, or exponential equation:
a. A pond is stocked with 200 adult bass, and the amount of bass increases by 15% each year. **Solution** This scenario can be modeled with an exponential equation because the population increases by a percent each year. For example, in the first year there would be $200(0.15) = 30$ new bass, bringing the total to 230. In the next year there would be $230(0.15) = 34.5$ new bass bringing the total to 264.5 bass, and so on. This illustrates an example of compound growth.	a. In the year 2009 there were 432 cases of violent crime per 100,000 people in the US, and this number was decreasing by 26 per year.
b. In 2015 there were 4.7 million households without cable TV, and that number was increasing by 100,000 per year. **Solution** This scenario can be modeled with a linear equation because the quantity increases by the same constant amount each year. We can express 100,000 as 0.1 million and model the scenario with the equation $y = 0.1x + 4.7$, where x is the number of years after 2015, and y is the number of households without cable tv in millions.	b. A post on social media goes viral and the number of likes per day increases rapidly. After a few days the popularity levels off and the number of likes per day starts to fall.
c. A new business experiences negative profit (cumulative profit decreases) while it establishes a customer base and increases its revenue. Eventually the cumulative profit reaches a minimum negative value and begins to increase towards positive values. **Solution** This scenario can be modeled with a quadratic equation because it describes the behavior of a parabola that opens upwards. The cumulative profit begins by decreasing, then reaches a minimum value (vertex), and then begins to increase.	c. The average 4-door sedan costs $21,000 today, and due to inflation that cost would increase by 2.1% each year.

How can exponential equations model growth?

Key Terms

Exponential Growth Model

Summary

A common form of **Exponential Growth Model** is the following:

$$F = P(1+r)^t$$

- P is the initial value
- r is the growth rate (usually expressed as a percent)
- t is the amount of time (usually in years)
- F is the future value, that is, the total amount after t time units have passed

Consider the previous example of investing $1,000 in an account earning 10% interest compounded annually. How much money would you have in the account after 5 years? We could compute this using $F = P(1+r)^t$:

$$F = 1000(1+0.10)^5 = 1000(1.1)^5 = 1610.51$$

After 5 years you would have $1,610.51 in the account.

Notice the exponential growth model has four variables in it. We could be given any of the three variables and be asked to find the fourth. Below we will investigate cases solving for F, P, and r.

Notes

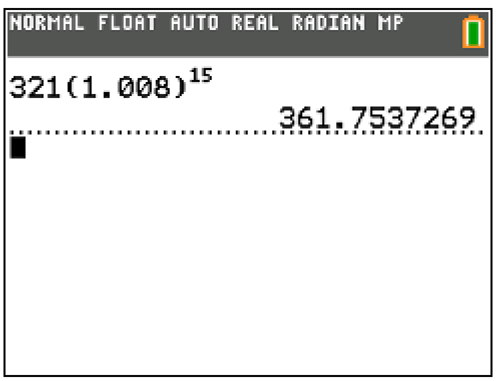
Guided Example 2

In 2015 the US population was 321 million and was growing at a rate of 0.8% each year. Use this information to create an exponential model and estimate the US population in the year 2030.

Solution We are given an initial population, a growth rate, and an amount of time, and we are asked to find the future value. We can summarize this information in the following way:

Given $P = 321$, $r = 0.008$, and $t = 15$ (2030-2015), find F. Substitute these values into the exponential growth model $F = P(1+r)^t$ and solve for F:

$$\begin{aligned} F &= 321(1+0.008)^{15} \\ &= 321(1.008)^{15} \\ &\approx 362 \end{aligned}$$



According to our model, in the year 2030 the US population should reach about 362 million people.

Practice

In 2015 the population of Japan was 127 million and had a growth rate of -0.1% each year. Use this information to create an exponential model and estimate the population of Japan in the year 2030.

Guided Example 3

How much should be invested in a savings account earning 5% interest compounded annually, if you want to have $10,000 after 6 years?

Solution We are given an interest rate, and an amount of time, and a future value, and we are asked to find the initial investment needed to attain that future value. We can summarize this information in the following way:

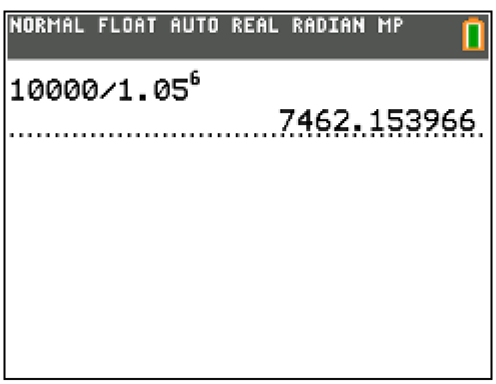
Given $r = 0.05$, $F = 10{,}000$, and $t = 6$, find P. We will substitute these values into the exponential growth model $F = P(1+r)^t$ and solve for P:

$10000 = P(1+0.05)^6$ Put in the values

$\frac{10000}{(1.05)^6} = \frac{P(1.05)^6}{(1.05)^6}$ Divide both sides by $(1.05)^6$

$\frac{10000}{(1.05)^6} = P$ Simplify the expression

$7462.15 \approx P$



You would need to invest about $7,462.15 in order to have $10,000 in 6 years.

One thing to note here is that if we rounded down to get 7462.15, we would have only $9,999.99 after 6 years Often calculations where we find present values should automatically be rounded up. So, a better answer would be $7,462.16.

Practice

A population of trout were seeded into a lake and increased by 60% per year. After 4 years there were 1500 trout in the lake. Use this information to create an exponential model and find the size of the initial population that was introduced to the lake.

Guided Example 4

A bacteria culture of 200 bacteria is started in a petri dish. After 4 hours the population has grown to 500 bacteria. Use this information to create an exponential model and estimate the growth rate of the bacteria culture.

Solution We are given an initial population, an amount of time, and a future value, and asked to find the growth rate. We can summarize this information in the following way:

Given $P = 200$, $F = 500$, and $t = 4$, find r. Substitute these values into the exponential growth model $F = P(1+r)^t$ and solve for r:

$$500 = 200(1+r)^4$$
Put in values

$$\frac{500}{200} = \frac{200(1+r)^4}{200}$$
Divide both sides by 200

$$\frac{5}{2} = (1+r)^4$$
Simplify

$$\sqrt[4]{\frac{5}{2}} = \sqrt[4]{(1+r)^4}$$
Undo the power with a fourth root.

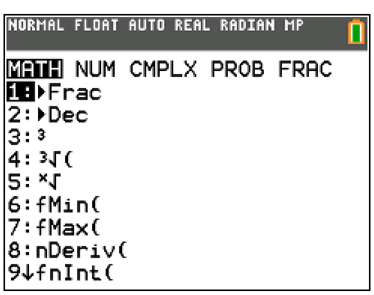
$$\sqrt[4]{\frac{5}{2}} = 1+r$$

$$\sqrt[4]{\frac{5}{2}} - 1 = 1 + r - 1$$
Subtract 1 to isolate r

$$1.2574 - 1 = r$$
Compute the root on your calculator

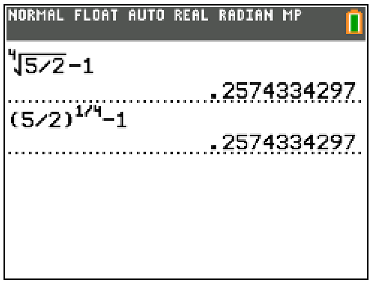
$$0.257 \approx r$$

Note that the $\sqrt[x]{}$ command is found in the MATH menu on a graphing calculator.



Practice

Suppose your parents placed $5,000 in a college savings account when you were born, and when you turned 18 the account had grown to $22,000. Assuming the interest was compounded annually, what was the interest rate on the account?

This root may also be thought of as a ¼ power and evaluate with the ^ button.  Since r is a growth rate, we write our final answer as a percent: $r = 25.7\%$. So according to our model, the population of bacteria increased by about 25.7% each hour.	

How are exponential equations solved using logarithms?

Key Terms

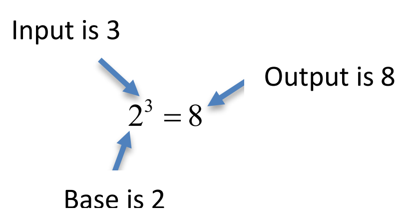
Logarithm

Common Logarithm

Exponent Property of Logarithms

Summary

We have yet to solve an exponential model for time, *t*. Because *t* is in the exponent position, it requires the use of a special algebraic function called a logarithm. A **logarithm** is the inverse of an exponential. For example, given the exponential equation $2^3 = 8$, we could rewrite it as a logarithmic equation in the following form: $\log_2(8) = 3$. That is, $\log_2(8)$ represents the power that we raise 2 to, in order to get 8. We can think of this relationship as an input/output relationship to make this easier to understand.



Thinking this way, we say that the input to the base is 3 giving an output of 8. A logarithm reverses the role of the input and output. Now the base on the logarithm is 2. The input to the logarithm is 8 and the output is 3.

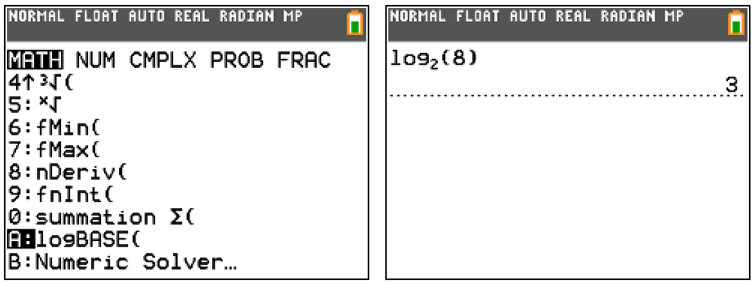
Input is 8

Output is 3

$$\log_2(8) = 3$$

Base is 2

You can use your calculator to verify this relationship by selecting the MATH button and then scrolling down to the LogBASE option.



We will often use a common logarithm, which is a base 10 logarithm, and is abbreviated as "log."

Common Logarithm:

$$\log_{10}(x) = \log(x)$$

There is a lot to say about logarithms, but we are going to use them in a very specific way that requires the following property:

Exponent Property of Logarithms:

$$\log(x^a) = a\log(x)$$

This property will allow us to extract a value from the exponent position. For example, consider the equation $2^x = 7$. We know that $2^2 = 4$ and that $2^3 = 8$, so the power we raise 2 to in order to get 7 must be between 2 and 3. To get a more exact result, we will apply a logarithm and use the exponent property:

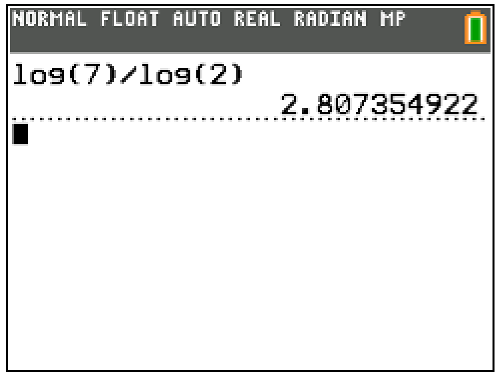
$$2^x = 7$$

$$\log(2^x) = \log(7)$$ Apply logs to each side of the equation

$$x\log(2) = \log(7)$$ Apply the exponent property of logarithms

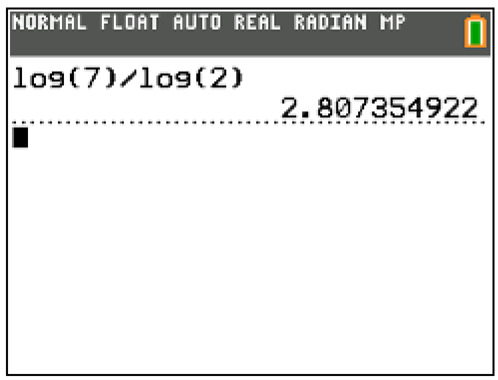
$$\frac{x\log(2)}{\log(2)} = \frac{\log(7)}{\log(2)}$$ Divide both sides by $\log(2)$

$$x \approx 2.807$$ Use a calculator to approximate to three decimal places



Notice $2^{2.807} \approx 6.998$ which is approximately 7.

Notice that we could have solved $2^x = 7$ by converting it to a logarithm directly. In this case we would get $x = \log_2(7)$. This value is exactly the same as $\frac{\log 7}{\log 2}$.



This relationship comes in handy if your calculator does not have a logBASE command. When this happens, we can use the relationship

$$\log_b a = \frac{\log a}{\log b}$$

to calculate the logarithm using the log button on the calculator.

Below we will apply these methods to exponential models to solve for *t*.

<u>Notes</u>

Guided Example 5

If $4500 is placed into an account earning 4.3% interest compounded annually, how long will it take for the amount in the account to double?

Solution If we want to double our initial amount of $4,500, then that means we are seeking a future amount of $9,000. We can summarize this information in the following way: Given P = 4500, r = 0.043, and F = 9000, find t. Substitute these values into the exponential growth model $F = P(1+r)^t$ and solve for t:

$$9000 = 4500(1+0.043)^t$$

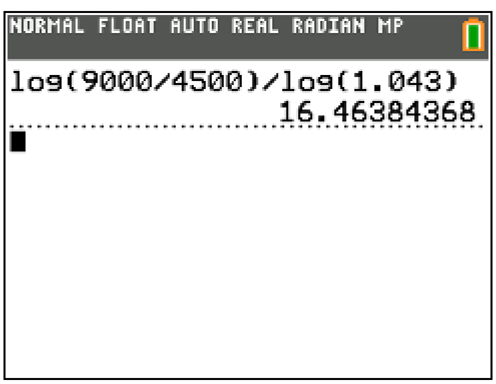
$$\frac{9000}{4500} = \frac{4500(1+0.043)}{4500}$$ Divide both sides by 4500

$$\log\left(\frac{9000}{4500}\right) = \log(1.043^t)$$ Take the log of both sides

$$\log\left(\frac{9000}{4500}\right) = t \cdot \log(1.043)$$ Use the exponential property of logs

$$\frac{\log\left(\frac{9000}{4500}\right)}{\log(1.043)} = t$$ Divide both sides by $\log(1.043)$

$$16.5 \approx t$$ Evaluate the log on a calculator



It would take approximately 16.5 years for our initial investment of $4,500 to double to $9,000.

Practice

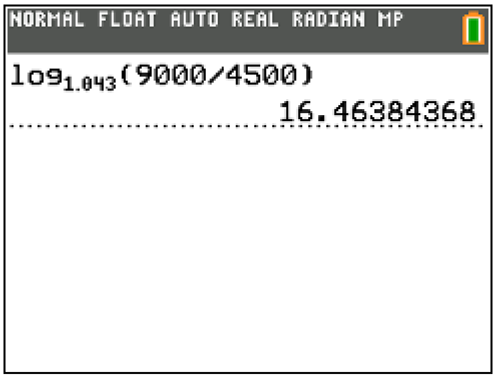
If you invest $2,500 in an account earning 5.4% compounded annually, and you want to save $20,000 dollars for a down payment on a house, how long must you wait until you have that down payment?

We can also solve the equation by converting directly to a logarithm.

$$9000 = 4500(1+0.043)^t$$

$$\frac{9000}{4500} = 1.043^t$$ Divide both sides by 4500

$$t = \log_{1.043}\left(\frac{9000}{4500}\right)$$ Convert to a logarithm



Guided Example 6	Practice
In 2015 the US population was 321 million and was growing at a rate of 0.8% each year. In what year will the population reach 400 million? **Solution** We are given an initial population, a growth rate, and a future population, and we must solve for time. We can summarize this information in the following way: Given P = 321, r = 0.008, and F = 400, find t. Substitute these values into the exponential growth model $F = P(1+r)^t$ and solve for t:	In 2015 the population of India was 1.225 billion and was growing at a rate of 1.2% each year. In what year will the population double?

$$400 = 321(1+0.008)^t$$

$$\frac{400}{321} = \frac{321(1+0.008)^t}{321}$$ Divide both sides by 321

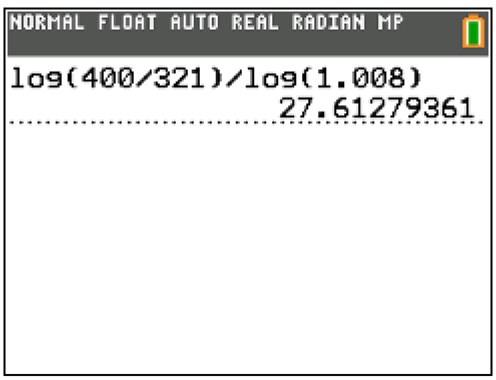
$$\log\left(\frac{400}{321}\right) = \log(1.008)^t$$ Apply log to both sides

$$\log\left(\frac{400}{321}\right) = t \cdot \log(1.008)$$ Apply the exponent property of logs

$$\frac{\log\left(\frac{400}{321}\right)}{\log(1.008)} = t \cdot \frac{\log(1.008)}{\log(1.008)}$$ Divide both sides by $\log(1.008)$

$$\frac{\log\left(\frac{400}{321}\right)}{\log(1.008)} = t$$ Evaluate logs on a calculator

$$28 \approx t$$

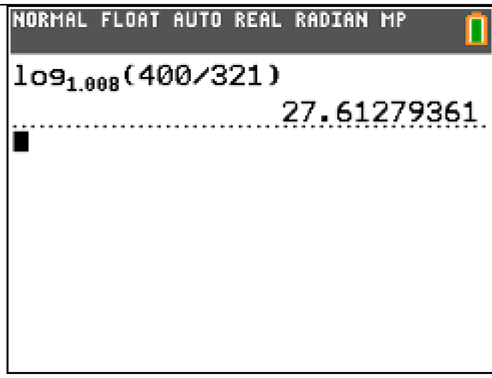


We can also convert to a logarithm as shown below.

$$400 = 321(1+0.008)^t$$

$$\frac{400}{321} = 1.008^t$$ Divide both sides by 321

$$t = \log_{1.008}\left(\frac{400}{321}\right)$$ Convert to a logarithm

 According to our model it the US population will reach 400 million people in approximately 28 years after 2015, or by the year 2043.	