Section 5.4 Annuities

- How do you calculate the future value of an ordinary annuity?
- What is a sinking fund?

How do you calculate the future value of an ordinary annuity?

Key Terms

Annuity Simple ordinary annuity

Term

Summary

A sequence of payments or withdrawals made to or from an account at regular time intervals is called an **annuity**. The **term** of the annuity is length of time over which the payments or withdrawals are made. There are several different types of annuities. An annuity whose term is fixed is called an annuity certain. An annuity that begins at a definite date but extends indefinitely is called a perpetuity. If an annuities term is not fixed, it is called a contingent annuity. Annuities that are created to fund a purchase at a later date like some equipment or a college education are called sinking funds.

The payments for an annuity may be made at the beginning or end of the payment period. In an **ordinary annuity**, the payments are made at the end of the payment period. If the payment is made at the beginning of the payment period, it is called an **annuity due**. In this text we'll only examine annuities in which the payment period coincides with the interest conversion period and the payments are made at the end of each period. This type of annuity is called a **simple ordinary annuity**.

Let's look at an ordinary annuity that is certain and simple. By this, we mean an annuity over a fixed term whose payment period matches the interest conversion period. Additionally, the payments to the annuity are made at the end of the payment period. Suppose a payment of $1000 is made semiannually to the annuity over a term of three years. If the annuity earns 4% per year compounded semiannually, the payment made at the end of the first six-month period will accumulate

$$1000\left(1+\tfrac{.04}{2}\right)^5$$

This means $1000 is multiplied by 1.02 five times, once for each of the remaining six-month periods.

The next payment also earns interest, but over 4 six-month periods. This payment has a future value of

$$1000\left(1+\tfrac{.04}{2}\right)^4$$

This process continues until we have the future value for each payment.

First payment period	\$1000 grows to $\$1000\left(1+\tfrac{.04}{2}\right)^5 \approx \$1104.08$
Second payment period	\$1000 grows to $\$1000\left(1+\tfrac{.04}{2}\right)^4 \approx \$1082.43$
Third payment period	\$1000 grows to $\$1000\left(1+\tfrac{.04}{2}\right)^3 \approx \$1061.21$
Fourth payment period	\$1000 grows to $\$1000\left(1+\tfrac{.04}{2}\right)^2 \approx \$1040.40$
Fifth payment period	\$1000 grows to $\$1000\left(1+\tfrac{.04}{2}\right)^1 \approx \$1020.00$
Sixth payment period	\$1000

The last payment occurs at the end of the last period and earns no interest. The sum of these amounts is

$$FV = 1000+1000(1.02)+1000(1.02)^2+1000(1.02)^3+1000(1.02)^4+1000(1.02)^5 \approx 6308.12$$

You can add the amount from each period to find the future value of the annuity, but this becomes tedious when there are many annuity payments. In this case, you can use the annuity formula to find the future value of the annuity.

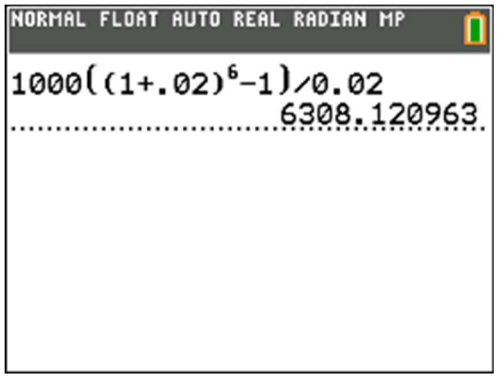
Future Value of an Ordinary Annuity

If equal payments of R are made into an ordinary annuity for n periods at an interest rate of i per period, the future value of the annuity F is

$$F = R\left[\frac{(1+i)^n-1}{i}\right]$$

For the ordinary annuity above, a payment of $1000 is made semiannually to the annuity over a term of three years. If the annuity earns 4% per year compounded semiannually, the future value is

$$F = 1000\left[\frac{(1+0.02)^6 - 1}{0.02}\right] \approx 6308.12$$



Note that *i* is the interest rate per period. This means the annual interest rate must be divided by 2 since the payments are made semiannually (twice per year).

Notes

Guided Example 1

An investor deposits \$500 in a simple ordinary annuity at the end of each six-month payment period. This annuity earns 10% per year, compounded semiannually.

a. Find the future value if payments are made for three years.

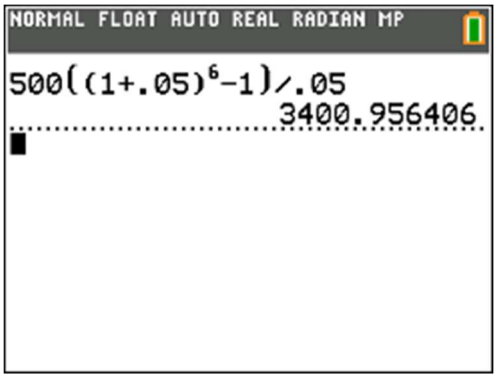
Solution Find the future value of this ordinary annuity using

$$F = R\left[\frac{(1+i)^n - 1}{i}\right]$$

In this case, $R = 500$, $i = \frac{0.10}{2} = 0.05$, and $n = 6$. This gives

$$F = 500\left[\frac{(1+.05)^6 - 1}{.05}\right] \approx 3400.96$$

This is calculated in a TI Graphing Calculator as shown below.



The six payments of \$500 have earned \$3400.96 – \$3000 or \$400.96 in interest over the life of the annuity.

Practice

An employee deposits \$100 in a simple ordinary annuity monthly. This annuity earns 8% per year, compounded monthly.

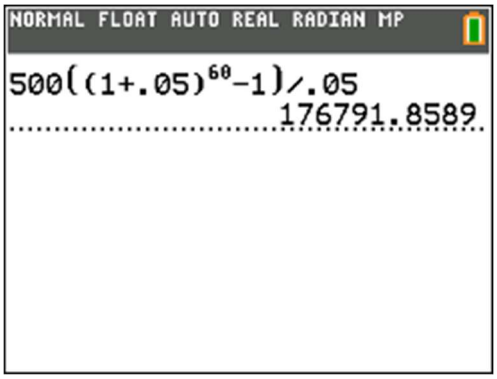
a. Find the future value if payments are made for ten years.

b. Find the future value if payments are made for 30 years.

Solution In this ordinary annuity, the term is much longer. Set $R = 500$, $i = 0.05$, and $n = 30 \cdot 2 = 60$ in the formula for the future value of an annuity, we get

$$F = 500\left[\frac{(1+.05)^{60}-1}{.05}\right] \approx 176,791.86$$

This is calculated in a TI Graphing Calculator as shown below.



c. How much interest is earned over the 30-year term in part b?

Solution Over the term of the annuity, sixty payments of $500 are made for a total of $30,000. This yields $176,791.86 – $30,000 or $146,791.86 in interest.

b. Find the future value if payments are made for 35 years.

c. How much interest is earned over the 35-year term in part b?

Notes

What is a sinking fund?

Key Terms

Sinking fund

Summary

Annuities that are created to fund a purchase at a later date like some equipment or a college education are called sinking funds. In a sinking fund, the future value is known and another quantity in the annuity formula,

$$F = R\left[\frac{(1+i)^n - 1}{i}\right]$$

is being solved for. In the example below, a value for F is given and the payment R is calculated that leads to that future value.

Notes

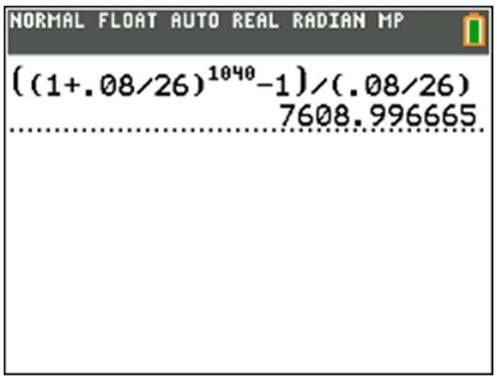
Guided Example 2

Suppose you want to accumulate $2,000,000 in a retirement account in 40 years. The retirement account averages an interest rate of 8% per year. How much would you need to deposit every two weeks (directly from your paycheck) to accumulate $2,000,000?

Solution Since deposits are being made at the end of each two week period, this is an ordinary annuity where the future value is $F = 2000000$, the interest rate per period is $i = \frac{0.08}{26}$, and the number of periods is $n = 26 \cdot 40$ or 1040. Put the values into the ordinary annuity formula,

$$2000000 = R\left[\frac{\left(1+\frac{0.08}{26}\right)^{1040}-1}{\frac{0.08}{26}}\right]$$

and work out the quantity in brackets on the right-hand side:



Putting this value into the equation gives

$$2000000 = R \cdot 7608.996665$$

Now divide each side by 7608.996665 to get the payment R,

$$R = \frac{2000000}{7608.996665} \approx 262.85$$

Each payment would need to be approximately $262.85 to accumulate $2,000,000.

Practice

Suppose you want to have $25,000 in an account in 6 years to purchase a new vehicle. The account earns 3.25% per year. How much would you need to put into the account each month to accumulate $25,000?