# Section 5.5 Amortization

- How do you calculate the present value of an annuity?
- How do you find the payment to pay off an amortized loan?
- What is an amortization schedule?

## How do you find the present value of an annuity?

Key Terms

Present value

Summary

We have been using the ordinary annuity formula,

$$F = R\left[\frac{(1+i)^n - 1}{i}\right]$$

to find the future value of payments made to an annuity. Often, we would like to know how much we would need to deposit all at once with compound interest to obtain the same future value.as making payments to an annuity. In this case, we want to know when the future value from compound interest,

$$F = P(1+i)^n$$

is equal to the future of the annuity.

$$F = R\left[\frac{(1+i)^n - 1}{i}\right]$$

By setting the two right-hand sides equal to each other, we can determine the present value *P* of the annuity. This leaves us with the equation,

$$P(1+i)^n = R\left[\frac{(1+i)^n - 1}{i}\right]$$

If we know the interest rate per period *i*, the payment *R*, and the number of periods *n*, we can solve for the present value *P* as illustrated in the examples below.

Guided Example 1

Find the present value of an ordinary annuity with payments of \$10,000 paid semiannually for 15 years at 5% compounded semiannually.

**Solution** We'll use the formula

$$P(1+i)^n = R\left[\frac{(1+i)^n - 1}{i}\right]$$

to find the present value *P*. From the problem statement, we know that

$$R = 10000$$
$$i = \frac{0.05}{2} = 0.025$$
$$n = 15 \cdot 2 = 30$$

Put these values into the formula and solve for PV:

$$P(1+0.025)^{30} = 10000\left[\frac{(1+0.025)^{30} - 1}{0.025}\right]$$ Work out the expression on each side

$$P \cdot 2.097567579 \approx 439027.0316$$

$$P \approx \frac{439027.0316}{2.097567579}$$ Isolate *P* using division.

$$P \approx 209302.93$$

This means that if we deposit \$209,302.93 with compound interest or deposit \$10,000 semiannually for 15 years, we will end up with the same future value of \$439,027.03 (the number in blue from the annuity formula).

Practice

Find the present value of an ordinary annuity with payments of \$90,000 paid annually for 25 years at 8% compounded annually.

## How do you find the payment to pay off an amortized loan?

Key Terms

Payment Amortization

Summary

Auto or home loans are often made to consumers so that they can afford a large purchase. In these types of loans, some amount of money is borrowed. Fixed payments are made to pay off the loan as well as any accrued interest. This process is called amortization.

In the language of finance, a loan is said to be amortized if the amount of the loan and interest are paid using fixed regular payments. From the perspective of the lender, the amount borrowed needs to be paid back with compound interest. From the perspective of the borrower, the amount borrowed, and interest is paid back via payments in an ordinary annuity:

Future Value with Compound Interest

Future Value of Ordinary Annuity

$$\underbrace{P(1+i)^n}_{\text{Lender: Wants amount borrowed } P \text{ and interest}} = \underbrace{R\left[\frac{(1+i)^n - 1}{i}\right]}_{\text{Borrower: Pay back amount borrowed and interest with payments } R}$$

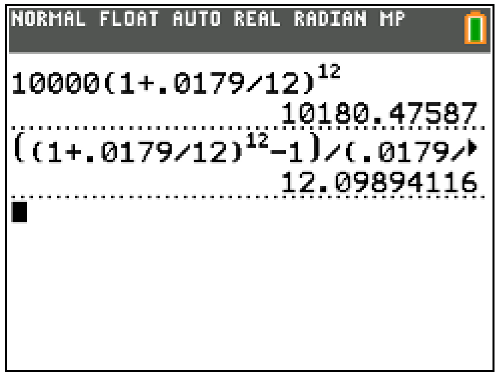
Suppose you want to borrow \$10,000 for an automobile. Navy Federal Credit Union offers a loan at an annual rate of 1.79% amortized over 12 months. To find the payment, identify the key quantities in the formula:

$$i = \tfrac{0.0179}{12}$$
$$P = 10,000$$
$$n = 12$$

Put these values into the payment formula to get

$$10000\left(1+\tfrac{0.0179}{12}\right)^{12} = R\left[\frac{\left(1+\tfrac{0.0179}{12}\right)^{12} - 1}{\tfrac{0.0179}{12}}\right]$$

Now work out the expressions on the left and the expression in the brackets.




$$10180.47587 \approx R \cdot 12.09894116$$
$$\frac{10180.47587}{12.09894116} \approx R$$
$$841.44 \approx R$$

The payment has been rounded up to the nearest cent. This ensures that the loan will be paid off. This means that you pay slightly more than is needed. In practice, this is accounted for in an amortization schedule (also called an amortization table).

Notes

<table>
<tr><th>Guided Example 2</th><th>Practice</th></tr>
<tr><td>
Find the payment necessary to amortize a loan of $7400 at an interest rate of 6.2% compounded semiannually in 18 semiannual payments.<br><br>
Solution To find the payment, use the formula<br><br>
$$P(1+i)^n = R\left[\frac{(1+i)^n - 1}{i}\right]$$<br>
In this case,<br><br>
$$P = 7400$$<br>
$$i = \frac{0.062}{2} = 0.031$$<br>
$$n = 18$$<br>
Put the values in the formula to give<br><br>
$$7400(1+0.031)^{18} = R\left[\frac{(1+0.031)^{18} - 1}{0.031}\right]$$<br>
Now work out the expression on each side and solve for R to give<br><br>
$$12819.99025 \approx R \cdot 23.62681015$$<br>
$$\frac{12819.99025}{23.62681015} \approx R$$<br>
$$542.61 \approx R$$<br>
This payment has been rounded up to the nearest cent. To find the total payments, multiply the amount of each payment by 18 to get<br><br>
$$542.61(18) = 9766.98$$<br>
To find the total amount of interest paid, subtract the original loan amount from the total payments,<br><br>
$$9766.98 - 7400 = 2366.98$$
</td><td>
Find the payment necessary to amortize a loan of $25,000 at an interest rate of 8.4% compounded quarterly in 24 quarterly payments.
</td></tr>
</table>

## What is an amortization schedule?

Key Terms

Amortization schedule

Summary

An amortization schedule (also called an amortization table) records the portion of the payment that applies to the principal and the portion that applies to interest. Using this information, we can determine exactly how much is owed on the loan at the end of any period.

The amortization schedule generally has 5 columns and rows corresponding to the initial loan amount and the payments. The heading for each column are shown below.

| Payment Number | Amount of Payment | Interest in Payment | Amount in Payment Applied to Balance | Outstanding Balance at the End of the Period |
|---|---|---|---|---|

To fill out the table, you need to carry out a sequence of steps to get each row of the table.

1. The first row of the table corresponds to the initial loan balance. Call this payment 0 and place the amount loaned in the column title "Outstanding Balance at the End of the Period".
2. Go to the next line in the table and enter the payment calculated on the loan.
3. In the same row, use $I = Prt$ to find the interest on the outstanding balance. Place this under the column titled "Interest in Payment".
4. To find the "Amount in Payment Applied to Balance", subtract the "Interest in the Payment" from the "Amount of Payment".
5. To find the new "Outstanding Balance at the End of the Period", subtract the "Amount in Payment Applied to Balance" from the "Outstanding Balance at the End of the Period" in the previous payment.

Fill out these quantities for all payments until the past payment. In the last payment, start by paying off the loan by making "Amount in Payment Applied to Balance" equal to the "Outstanding Balance at the End of the Period" in the second to last payment. This means the loan will be paid off resulting in the "Outstanding Balance at the End of the Period" for the final payment being 0. Finally, calculate the interest in the final payment and add it to the "Amount in Payment Applied to Balance" to give the final payment. Because of rounding in the payment, this may be slightly higher of lower than the other payments.

Let's look at an example of a $10,000 for an automobile. Navy Federal Credit Union offers a loan at an annual rate of 1.79% amortized over 12 months. The amortization schedule below shows the calculation of the quantities for payment 1 and the last payment. Other payments follow a similar process.

**3.**

$$I = Prt = 10000 \cdot \frac{.0179}{12} \cdot 1$$

Rounded to nearest cent

| Payment Number | Amount of Payment | Interest in the Payment | Amount in Payment Applied to Balance | Outstanding Balance at the End of the Period |
|---|---|---|---|---|
| 0 | | | | 10000 |
| 1 | 841.44 | 14.92 | 826.52 | 9173.48 |
| 2 | 841.44 | 13.68 | 827.76 | 8345.72 |
| 3 | 841.44 | 12.45 | 828.99 | 7516.73 |
| 4 | 841.44 | 11.21 | 830.23 | 6686.50 |
| 5 | 841.44 | 9.97 | 831.47 | 5855.03 |
| 6 | 841.44 | 8.73 | 832.71 | 5022.32 |
| 7 | 841.44 | 7.49 | 833.95 | 4188.37 |
| 8 | 841.44 | 6.25 | 835.19 | 3353.18 |
| 9 | 841.44 | 5.00 | 836.44 | 2516.74 |
| 10 | 841.44 | 3.75 | 837.69 | 1679.05 |
| 11 | 841.44 | 2.50 | 838.94 | 840.11 |
| 12 | 841.36 | 1.25 | 840.11 | 0.00 |

**1.** Starting balance

**2.** Calculated Payment

**4.** 841.44 – 14.92

**5.** 10000 – 826.52

1.25 + 840.11

Need to pay off the loan in the last payment

Note that the interest has been rounded to the nearest cent. Different lenders may round the interest in different ways. Make sure you understand the rounding for the interest and the payment in order to obtain the corresponding amortization schedule.

Guided Example 3

Suppose a loan of \$2500 is made to an individual at 6% interest compounded quarterly. The loan is repaid in 6 quarterly payments.

a. Find the payment necessary to amortize the loan.

**Solution** To find the payment on the loan, use the formula

$$P(1+i)^n = R\left[\frac{(1+i)^n - 1}{i}\right]$$

For this problem, the interest rate per period is $i = \frac{0.06}{4}$. The present value is $P = 2500$ and the number of periods is $n = 6$. Using these values gives

$$2500\left(1+\tfrac{0.06}{4}\right)^6 = R\left[\frac{\left(1+\frac{0.06}{4}\right)^6 - 1}{\frac{0.06}{4}}\right]$$

$$2733.60816 \approx R \cdot 6.22955093$$

$$\frac{2733.60816}{6.22955093} \approx R$$

$$438.813 \approx R$$

Depending on how the rounding is done, this gives a payment of \$438.81 or 438.82. For a calculated payment, the payment is often rounded to the nearest penny. However, many finance companies will round up to insure the last payment is no more than the other payments.

b. Find the total payments and the total amount of interest paid based on the calculated monthly payments.

**Solution** The total payments (assuming the payment is rounded to the nearest penny) is

$$438.81(6) = 2632.86$$

The total amount of interest is

$$2632.86 - 2500 = 132.86$$

c. Find the total payments and the total amount of interest paid based on an amortization table.

**Solution** Making the amortization table takes several steps. Let me take it in pieces using the payment from above.

| Payment Number | Amount of Payment | Interest in Payment | Amount in Payment Applied to Balance | Outstanding Balance at the End of the Period |
|---|---|---|---|---|
| 0 | | | | 2500 |
| 1 | 438.81 | 37.50 | 401.31 | 2098.69 |

$2500 \cdot \frac{0.06}{4}$ $\quad 438.81 - 37.5$ $\quad 2500 - 401.31$

The next row is filled out in a similar manner.

| Payment Number | Amount of Payment | Interest in Payment | Amount in Payment Applied to Balance | Outstanding Balance at the End of the Period |
|---|---|---|---|---|
| 0 | | | | 2500 |
| 1 | 438.81 | 37.50 | 401.31 | 2098.69 |
| 2 | 438.81 | 31.48 | 407.33 | 1691.36 |

$2098.69 \cdot \frac{0.06}{4}$ $\quad 438.81 - 31.48$ $\quad 2098.69 - 407.33$

Continue this process until the last row

| Payment Number | Amount of Payment | Interest in Payment | Amount in Payment Applied to Balance | Outstanding Balance at the End of the Period |
|---|---|---|---|---|
| 0 | | | | 2500 |
| 1 | 438.81 | 37.50 | 401.31 | 2098.69 |
| 2 | 438.81 | 31.48 | 407.33 | 1691.36 |
| 3 | 438.81 | 25.37 | 413.44 | 1277.92 |
| 4 | 438.81 | 19.17 | 419.64 | 850.28 |
| 5 | 438.81 | 12.87 | 425.94 | 432.34 |
| 6 | | | | |

After the fifth payment, we have $432.34 of principal left to pay in the final payment. So, this is the principal in the sixth payment. The interest is found by paying interest on the outstanding balance,

$$432.34 \cdot \frac{0.06}{4} \approx 6.49$$

This gives a final payment of

$$432.34 + 6.49 = 438.83$$

Now put these numbers into the amortization table.

| Payment Number | Amount of Payment | Interest in Payment | Amount in Payment Applied to Balance | Outstanding Balance at the End of the Period |
|---|---|---|---|---|
| 0 | | | | 2500 |
| 1 | 438.81 | 37.50 | 401.31 | 2098.69 |
| 2 | 438.81 | 31.48 | 407.33 | 1691.36 |
| 3 | 438.81 | 25.37 | 413.44 | 1277.92 |
| 4 | 438.81 | 19.17 | 419.64 | 850.28 |
| 5 | 438.81 | 12.87 | 425.94 | 432.34 |
| 6 | 438.83 | 6.49 | 432.34 | 0 |

Since the payments had been rounded to the nearest penny (rounded down), the final payment is slightly higher than the previous payments. Adding all the payments we get a total of $2632.88. Adding the interest amounts gives total interest of $132.88.

Practice

Suppose a loan of $5000 is made to an individual at 4% interest compounded semiannually. The loan is repaid in 6 semiannual payments.

a. Find the payment necessary to amortize the loan. Round the payment to the nearest penny.

b. Find the total payments and the total amount of interest paid based on the calculated monthly payments.

c. Find the total payments and the total amount of interest paid based on an amortization table.

| Payment Number | Amount of Payment | Interest in Payment | Amount in Payment Applied to Balance | Outstanding Balance at the End of the Period |
|---|---|---|---|---|
| 0 | | | | |
| 1 | | | | |
| 2 | | | | |
| 3 | | | | |
| 4 | | | | |
| 5 | | | | |
| 6 | | | | |

**Chapter 5 Practice Solutions**

Section 5.1

1. 48
2. 10
3. 40%
4. 300
5. About 2.8%
6. About -2.5%
7. a. $11342.35, b. $.17418.02
8. 1.40%

Section 5.2

1. a. $1488, b. $288
2. Approximately 1.32 years
3. Approximately 11.1%
4. a. $45460, b. $5460, c. $591, d. $709.20
5. $6628.92
6. Rate per period is about 2.24% and the annual rate is about 8.96%
7. $12561.52
8. About 104.32 periods or 8.69 years

Section 5.3

1. $4.58
2. Average daily balance is $764.23 and the corresponding finance charge is $13.19
3. Average daily balance is $307.68 and the corresponding finance change is $5.49

Section 5.4

1. a. $18294.60, b. $229388.25, c. Payments total to $42000 and interest totals to $187388.25
2. $314.94

Section 5.5

1. $960729.60 to nearest cent
2. $1336.80 to nearest cent
3. a. 892.63 to nearest cent, b. Total payments are 5355.78 with total interest 355.78 d. All numbers to nearest penny

| Payment Number | Amount of Payment | Interest in Payment | Amount in Payment Applied to Balance | Outstanding Balance at the End of the Period |
|---|---|---|---|---|
| 0 | | | | 5000 |
| 1 | 892.63 | 100.00 | 792.63 | 4207.37 |
| 2 | 892.63 | 84.15 | 808.48 | 3398.89 |
| 3 | 892.63 | 67.98 | 824.65 | 2574.24 |
| 4 | 892.63 | 51.48 | 841.15 | 1733.09 |
| 5 | 892.63 | 34.66 | 857.97 | 875.12 |
| 6 | 892.62 | 17.50 | 875.12 | 0 |

Total payments are 5355.77.

Total interest is 355.77.