Section 10.3 Computing Limits Algebraically

Question 1 How can a limit be computed algebraically?

Question 2 How do you evaluate limits involving difference quotients?

Question 1 – How can a limit be computed algebraically?

Key Terms

Algebraically	Rationalize
Common denominator	Conjugate

Summary

When attempting to compute a limit, a good starting strategy is to simply substitute the x value into the expression. If you get a number, then you are done. That is the value of the limit.

Often you will get a 0 in the denominator and a nonzero number in the numerator. When this happens, you need to resort to one-sided limits and use a graph or table to estimate the value of the limit.

Suppose you substitute the x value into the function and obtain a zero in the numerator and denominator. In this case, you should carry out some algebra to see if the expression can be reduced. This may mean factoring, getting a common denominator, or rationalizing part of the fraction. Once the expression is simplified, you can substitute the x value to obtain the value of the limit.

Notes

Guided Example	Practice
Compute the value of the limit $$\lim_{x \to 3} \frac{x^2 - 5x + 6}{x - 3}$$ **Solution** If you substitute x = 3 into the numerator and denominator of the fraction, both are equal to zero. This indicates that there must be some type of algebra that we can do to simplify the fraction. Let's try factoring: $$\lim_{x \to 3} \frac{x^2 - 5x + 6}{x - 3} = \lim_{x \to 3} \frac{(x-2)\cancel{(x-3)}}{\cancel{x-3}} = \lim_{x \to 3} x - 2$$ The factor that reduces is the factor that is causing the top and bottom to be zero. Without those factors we can substitute *x* = 3 into *x* - 2 to give $$\lim_{x \to 3} \frac{x^2 - 5x + 6}{x - 3} = 1$$	1. Compute the value of the limit $$\lim_{x \to 4} \frac{x^2 - 2x - 8}{x - 4}$$

Guided Example

Compute the value of the limit

$$\lim_{x\to 0}\frac{\frac{1}{x-6}+\frac{1}{6}}{x}$$

Solution When we substitute $x = 0$ into the expression, we get $\frac{0}{0}$ so there must be some algebra we can try to simplify the expression. In this case, we will get a common denominator of $6(x-6)$ and simplify:

$$\lim_{x\to 0}\frac{\frac{1}{x-6}+\frac{1}{6}}{x}=\lim_{x\to 0}\frac{\frac{6}{6(x-6)}+\frac{x-6}{6(x-6)}}{x}$$

$$=\lim_{x\to 0}\frac{\frac{6+x-6}{6(x-6)}}{x}$$ Combine fractions

$$=\lim_{x\to 0}\frac{\frac{x}{6(x-6)}}{x}$$ Add like terms

$$=\lim_{x\to 0}\frac{\cancel{x}}{6(x-6)}\cdot\frac{1}{\cancel{x}}$$ Invert and multiply

$$=\lim_{x\to 0}\frac{1}{6(x-6)}$$

Now that we have reduced the fraction, we can substitute $x = 0$ into the reduced expression to yield

$$\lim_{x\to 0}\frac{\frac{1}{x-6}+\frac{1}{6}}{x}=-\frac{1}{36}$$

Practice

2. Compute the value of the limit

$$\lim_{x\to 0}\frac{\frac{1}{4}-\frac{1}{x+4}}{x}$$

<u>Guided Example</u>

Compute the value of the limit

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}$$

Solution As with earlier examples, substituting x = 1 leads to $\frac{0}{0}$. To simplify this fraction, multiply the top and bottom by the conjugate of the top, $\sqrt{x}+1$:

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}=\lim_{x\to 1}\frac{\left(\sqrt{x}-1\right)}{\left(x-1\right)}\cdot\frac{\left(\sqrt{x}+1\right)}{\left(\sqrt{x}+1\right)}$$

$$=\lim_{x\to 1}\frac{x-1\sqrt{x}+1\sqrt{x}-1}{\left(x-1\right)\left(\sqrt{x}+1\right)}$$

Foil the top

$$=\lim_{x\to 1}\frac{\cancel{x-1}}{\cancel{\left(x-1\right)}\left(\sqrt{x}+1\right)}$$

Combine like terms

$$=\lim_{x\to 1}\frac{1}{\sqrt{x}+1}$$

Now that the expression has been reduced, substitute *x* = 0 into the reduced expression to give

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}=\frac{1}{2}$$

<u>Practice</u>

3. Compute the value of the limit

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$$

Question 2 – How do you evaluate limits involving difference quotients?

Key Terms

Difference quotient

Summary

Difference quotients have the form $\frac{f(a+h)-f(a)}{h}$. When they are used in limits, they typically lead to $\frac{0}{0}$. To evaluate a limit with a difference quotient, we need to simplify as in the previous question and substitute the value into the simplified expression.

Notes

Guided Example	Practice
Compute $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ where $f(x)=4x+3$ and $a=1$. **Solution** Start by evaluating $f(a)$ and $f(a+h)$ for the given function: $f(1)=4(1)+3$ $=7$ $f(1+h)=4(1+h)+3$ $=7+4h$ Put these values into the difference quotient and simplify: $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{7+4h-7}{h}$ $= \lim_{h\to 0} \frac{4h}{h}$ $= \lim_{h\to 0} 4$ Since there is no place to substitute h = 0 into the reduced difference quotient, $\lim_{h \to 0} \frac{f(1+h)-f(1)}{h} = 4$	1. Compute $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ where $f(x)=8x-1$ and $a=-2$.

Guided Example	Practice
Compute $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ where $f(x)=x^2-4$ and $a=3$.	2. Compute $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ where $f(x)=x^2-1$ and $a=2$.

Solution Start by evaluating $f(a)$ and $f(a+h)$ for the given function:

$$\begin{aligned} f(3) &= 3^2 - 4 \\ &= 5 \\ f(3+h) &= (3+h)^2 - 4 \\ &= 9 + 6h + h^2 - 4 \\ &= 5 + 6h + h^2 \end{aligned}$$

Put these values into the difference quotient and simplify:

$$\begin{aligned} \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0} \frac{5+6h+h^2-5}{h} \\ &= \lim_{h\to 0} \frac{6h+h^2}{h} && \text{Combine like terms} \\ &= \lim_{h\to 0} \frac{\cancel{h}(6+h)}{\cancel{h}} && \text{Factor and reduce} \\ &= \lim_{h\to 0} 6+h \end{aligned}$$

Substitute $h=0$ into the reduced difference quotient,

$$\lim_{h \to 0} \frac{f(3+h)-f(3)}{h} = 6$$

Guided Example	Practice
Compute $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ where $f(x) = x^2 - x$ **Solution** Start by evaluating $f(x+h)$ for the given function: $f(x+h) = (x+h)^2 - (x+h)$ $= x^2 + 2xh + h^2 - x - h$ Put this expression into the difference quotient with $f(x) = x^2 - x$ and simplify: $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-x-h-x^2+x}{h}$ $= \lim_{h\to 0} \frac{2xh+h^2-h}{h}$ Combine like terms $= \lim_{h\to 0} \frac{\cancel{h}(2x+h-1)}{\cancel{h}}$ Factor and reduce $= \lim_{h\to 0} 2x+h-1$ Substitute $h = 0$ into the reduced difference quotient, $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = 2x - 1$ The limit comes out as a function of x since we don't have specific values like earlier examples.	3. Compute $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ where $f(x) = x^2 + 2x$.

Guided Example	Practice
Compute $\lim\limits_{h \to 0} \dfrac{f(x+h)-f(x)}{h}$ where $f(x)=\dfrac{1}{x}$. **Solution** Start by evaluating $f(x + h)$ for the given function: $$f(x+h)=\frac{1}{x+h}$$ Put this expression into the difference quotient with $f(x)=\dfrac{1}{x}$ and simplify. To simplify you will need a common denominator of $x\left(x+h\right)$ $$\begin{aligned} \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h} \\ &= \lim_{h\to 0} \frac{\frac{x}{x\left(x+h\right)}-\frac{x+h}{x\left(x+h\right)}}{h} \\ &= \lim_{h\to 0} \frac{\frac{-h}{x\left(x+h\right)}}{h} \\ &= \lim_{h\to 0} \frac{-\cancel{h}}{x\left(x+h\right)}\cdot\frac{1}{\cancel{h}} \\ &= \lim_{h\to 0} \frac{-1}{x\left(x+h\right)} \end{aligned}$$ Substitute $h = 0$ into the reduced difference quotient, $$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \frac{-1}{x^2}$$	4. Compute $\lim\limits_{h \to 0} \dfrac{f(x+h)-f(x)}{h}$ where $f(x)=\dfrac{1}{2x}$.