Section 11.3 The Derivative at a Point

Question 1 – What is a derivative?

Question 2 – How do you compute the derivative at a point using a limit?

Question 3 - How can you use a tangent line to forecast function values?

Question 4 - What does the derivative at a point tell you about a function?

Question 1 – What is a derivative?

Key Terms

Derivative

Tangent line

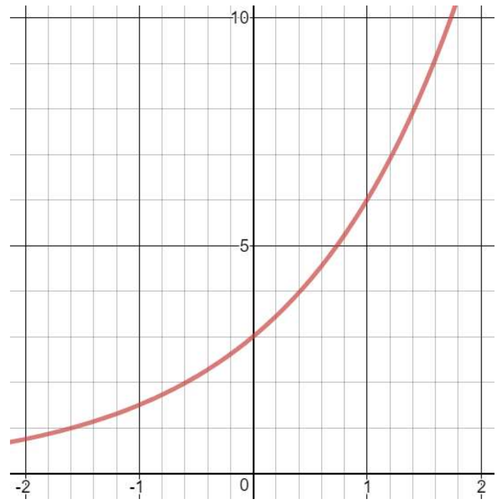
Summary

The derivative of a function $f(x)$ at a particular point $x = a$ is the slope of the tangent line to the function at $x = a$. To estimate the value of the derivative of f at a (denoted by $f'(a)$, draw the tangent line on a graph and then estimate its slope using any grids or scales available on the graph.

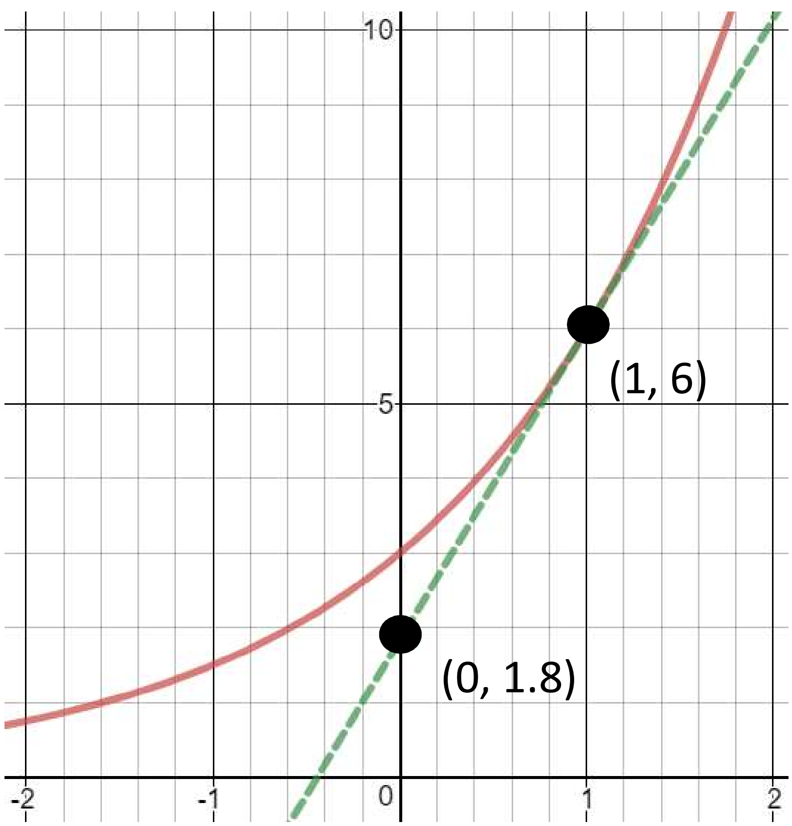
Notes

<u>Guided Example</u>

For the function $f(x)$ graphed below, estimate $f'(1)$



Solution Draw the tangent line on the graph and locate two points on the tangent line.

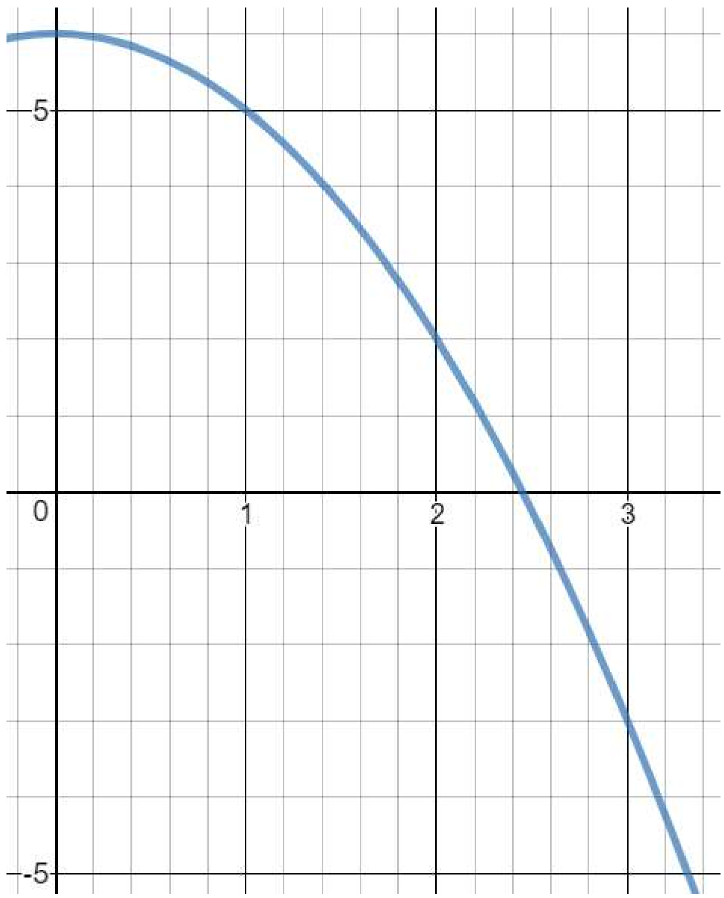


Calculate the slope from these two points to give an estimate of the value of the derivative,

$$f'(1) \approx \frac{6-1.8}{1.0} \approx 4.2$$

Practice

1. For the function $g(x)$ graphed below, estimate $g'(2)$



Question 2 – How do you compute the derivative at a point using a limit?

Key Terms

Limit definition of derivative at a point

Summary

The slope of the tangent line at a point may be calculated using the instantaneous rate of change at the point. For a function $f(x)$ at a point $x = a$, the derivative is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists. Evaluating this limit may require some of the tools from Section 10.3.

Notes

Guided Example	Practice
Suppose $f(x) = -2x - 3$. Use the definition of the derivative to compute $f'(8)$. **Solution** The definition of the derivative at a point $a = 8$ is, $$f'(8) = \lim_{h \to 0} \frac{f(8+h) - f(8)}{h} \quad .$$ To apply this definition, we need to calculate the two function values in the numerator of the difference quotient. $$f(8) = -2(8) - 3 = -19$$ $$f(8+h) = -2(8+h) - 3 = -19 - 2h$$ This is put into the definition to give $$f'(8) = \lim_{h \to 0} \frac{\cancel{-19} - 2h - (\cancel{-19})}{h}$$ $$= \lim_{h \to 0} \frac{-2h}{h}$$ $$= \lim_{h \to 0} -2$$ $$= -2$$ Since there is no h in the final limit, h approaching zero does nothing.	1. Suppose $f(x) = 5x - 3$. Use the definition of the derivative to compute $f'(-5)$.

Guided Example	Practice
Suppose $f(x)=3x^2-4x+1$. Use the definition of the derivative to compute $f'(2)$. **Solution** The definition of the derivative at a point $a=2$ is, $$f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}\ .$$ To apply this definition, we need to calculate the two function values in the numerator of the difference quotient. $$f(2)=3(2)^2-4(2)+1=5$$ $$\begin{aligned} f(2+h) &= 3(2+h)^2-4(2+h)+1 \\ &= 12+12h+3h^2-8-4h+1 \\ &= 3h^2+8h+5 \end{aligned}$$ Now put the function values into the difference quotient: $$\begin{aligned} f'(2) &= \lim_{h\to 0}\frac{3h^2+8h\cancel{+5}\cancel{-5}}{h} \\ &= \lim_{h\to 0}\frac{\cancel{h}(3h+8)}{\cancel{h}} \\ &= \lim_{h\to 0} 3h+8 \\ &= 8 \end{aligned}$$	Suppose $f(x)=2x^2+x-2$. Use the definition of the derivative to compute $f'(-1)$.

Question 3 – How can you use a tangent line to forecast function values?

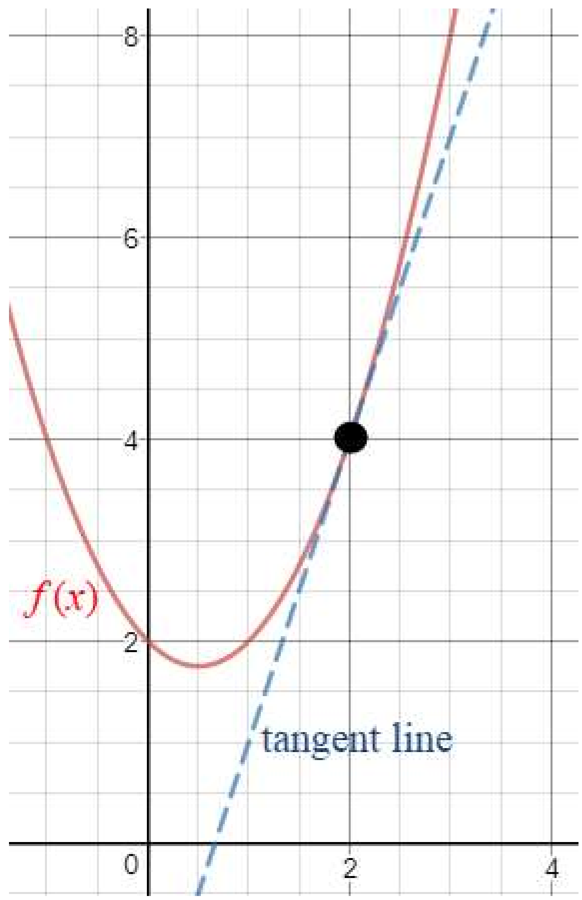
Key Terms

Tangent line

Forecast

Summary

The tangent line to a function can be used as a quick way to forecast the value of the function at some point. To find the equation of the tangent line, we start from $y = mx + b$. The slope of the tangent line, m, is found by evaluating the derivative at a given point. The vertical intercept, b, is found by substituting the point at which the line is tangent into the original function.



Notes

Guided Example	Practice
For the function $f(x)=x^2-x+2$, answer each of the questions below. a. Find the equation of the tangent line at $x = 2$. **Solution** To find the equation of the tangent line, we need to find the point on the function it will pass through and the slope of the tangent line at that point. To find the point on the function, simply substitute $x = 2$ into the function. $$f(2)=2^2-2+2=4$$ The tangent line will have to pass through the point (2, 4). To find the slope of the tangent line, we need to find the value of the derivative at $x = 2$: $$f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$ where $$f(2)=(2)^2-(2)+2=4$$ $$\begin{aligned} f(2+h)&=(2+h)^2-(2+h)+2\\ &=4+4h+h^2-2-h+2\\ &=h^2+3h+4\end{aligned}$$ Using these function values in the definition of the derivative gives $$\begin{aligned} f'(2)&=\lim_{h\to 0}\frac{h^2+3h\cancel{+4}\cancel{-4}}{h}\\ &=\lim_{h\to 0}\frac{\cancel{h}(h+3)}{\cancel{h}}\\ &=3\end{aligned}$$	1. For the function $f(x)=x^2-2x+3$, answer each of the questions below. a. Find the equation of the tangent line at $x = 1$.

Based on this information, the tangent line has slope 3 and passes through (2, 4). The equation is found by starting with the equation of a line, $y = mx + b$. We know the slope is 3 so we can let $m = 3$ to yield $$y = 3x + b$$ You can find the value of b by substituting (2, 4) into this equation. $$4 = 3(2) + b$$ $$-2 = b$$ Using this value, we find the equation of the tangent line, $$y = 3x - 2$$ b. Use the tangent line to forecast the value of the function at $x = 2.5$. **Solution** We can forecast values on the function using the tangent line. To do this, substitute x = 4 into the tangent line: $$y = 3(2.5) - 2 = 5.5$$	b. Use the tangent line to forecast the value of the function at $x = 1.75$.

Question 4 – What does the derivative at a point tell you about a function?

Key Terms

Units

Dependent variable

Independent variable

Summary

When you evaluate a derivative in the context of an application, you get a number that indicates how fast two quantities are changing with respect to each other. The units on the independent and dependent variable determine the units on the value of the derivative.

$$\text{units on the derivative} = \frac{\text{units of the dependent variable}}{\text{units on the independent variable}}$$

For instance, suppose we have a demand function $D(p)$. Since this is a demand function, it relates the unit price (in this case in dollars) to the number of units demanded at this price level. The derivative of this demand function would have units

$$\frac{\text{units}}{\text{dollars}}$$

Using this number, we can examine how an increase in price of 1 dollar will impact consumer demand.

Notes

Guided Example	Practice
The profit (in thousands of dollars) for selling x hundred units of compressors is $$P(x) = -4x^2 + 160x - 1000$$ Find and interpret $P'(10)$. **Solution** Start by writing out the definition for this derivative: $$P'(10) = \lim_{h \to 0} \frac{P(10+h) - P(10)}{h}$$ where $$P(10) = -4(10)^2 + 160(10) - 1000 = 200$$ $$\begin{aligned} P(10+h) &= -4(10+h)^2 + 160(10+h) - 1000 \\ &= -4(100 + 20h + h^2) + 160(10+h) - 1000 \\ &= -400 - 80h - 4h^2 + 1600 + 160h - 1000 \\ &= -4h^2 + 80h + 200 \end{aligned}$$ Using these function values in the definition of the derivative gives $$\begin{aligned} P'(10) &= \lim_{h \to 0} \frac{-4h^2 + 80h \cancel{+200} \cancel{-200}}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(-4h + 80)}{\cancel{h}} \\ &= 80 \end{aligned}$$ This number does not have much meaning without units attached to it. Since it represents a slope on the graph of $P(x)$, the corresponding units are the units of the dependent variable divided by the units of the independent variable. $$\frac{\text{thousands of dollars}}{\text{hundreds of compressors}}$$	1. The profit (in thousands of dollars) for selling x hundred units of calculators is $$P(x) = -5x^2 + 80x - 100$$ Find and interpret $P'(7)$.

Simplifying thousands divided by hundreds to give hundreds, these units become $$\frac{\text{tens of dollars}}{\text{compressors}}$$ Including this in our derivative gives $$P'(10) = 80\,\frac{\text{tens of dollars}}{\text{compressors}}$$ This means that increasing the number of compressors by 1 unit will increase the profit by 800 dollars. Another way to say this is to indicate that producing the 1001st compressor will increase profit by \$800.	