Section 11.4 The Derivative Function

Question 1 – What is a derivative function?

Question 2 – How do you calculate the derivative of a function from the definition?

Question 3 - What are the derivatives of some basic functions (linear, polynomial, power, exponential, and logarithmic)?

Question 1 – What is a derivative function?

<u>Key Terms</u>

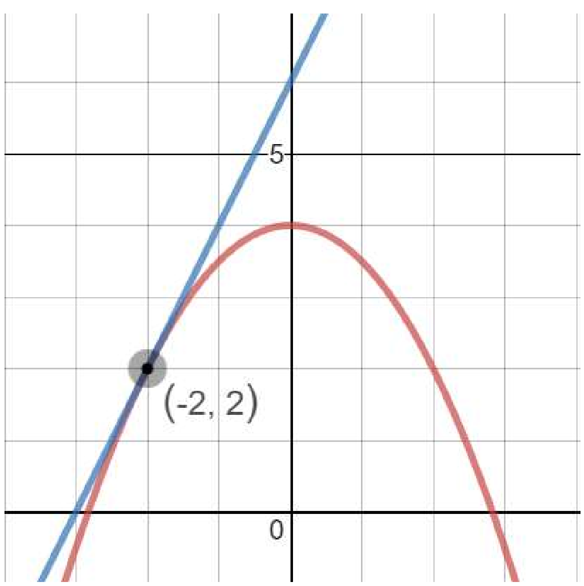
Derivative

Function

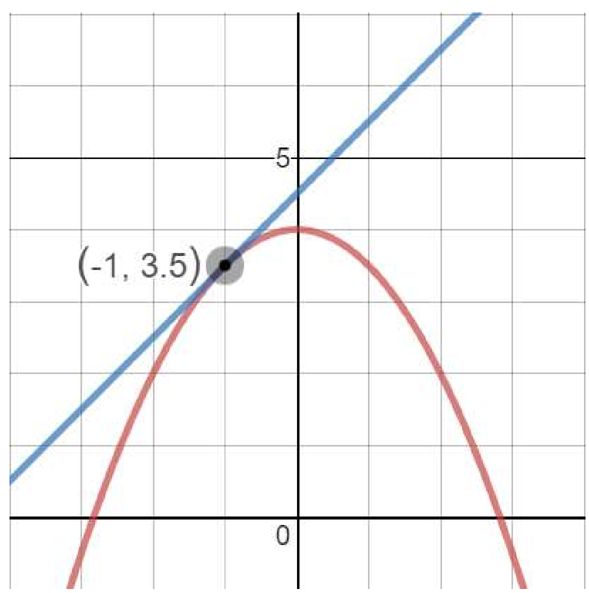
<u>Summary</u>

The derivative is a function whose outputs are equal to the slopes of a tangent line on a function. Using the geometric idea of the slope of a tangent line, we can select several points on a graph, draw a tangent line, and measure its slope. The resulting slope values may be graphed to obtain a graph of the derivative function.

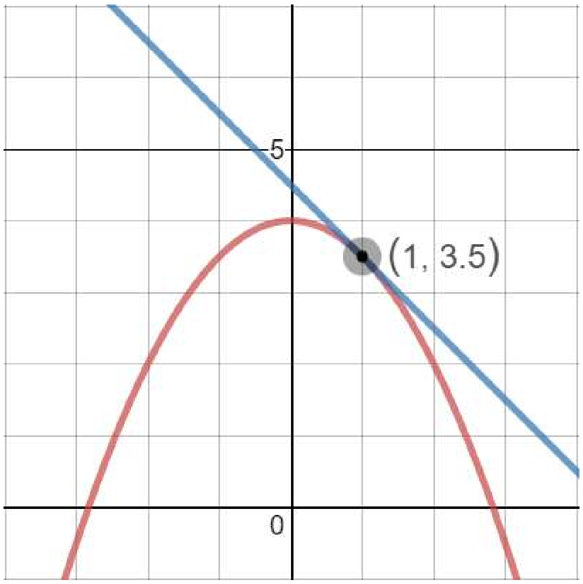
As an example, a parabola is graphed in red below with a tangent line in blue at x = -2, -1, 1, 2.



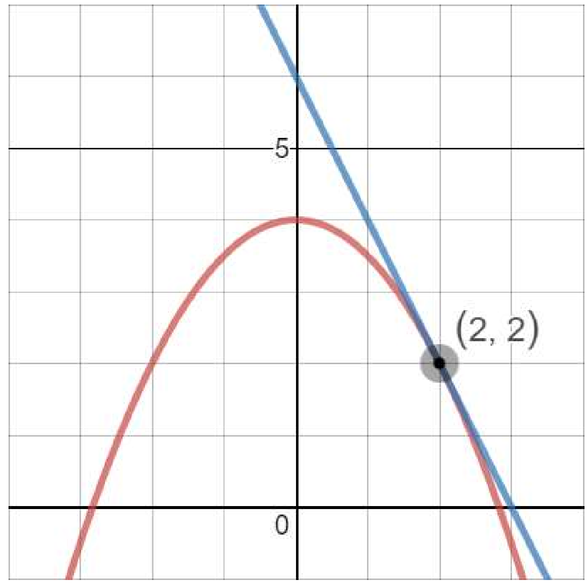
Slope of the tangent line at x = -2 is 2.



Slope of the tangent line at x = -1 is 1.



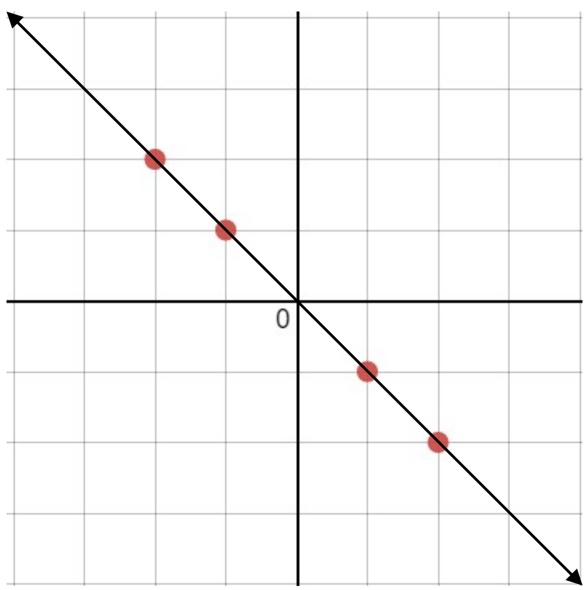
Slope of the tangent line at $x = 1$ is -1.



Slope of the tangent line at $x = 2$ is -2.

Let's put these values into a table and graph them:

x	$f'(x)$
-2	2
-1	1
1	-1
2	-2

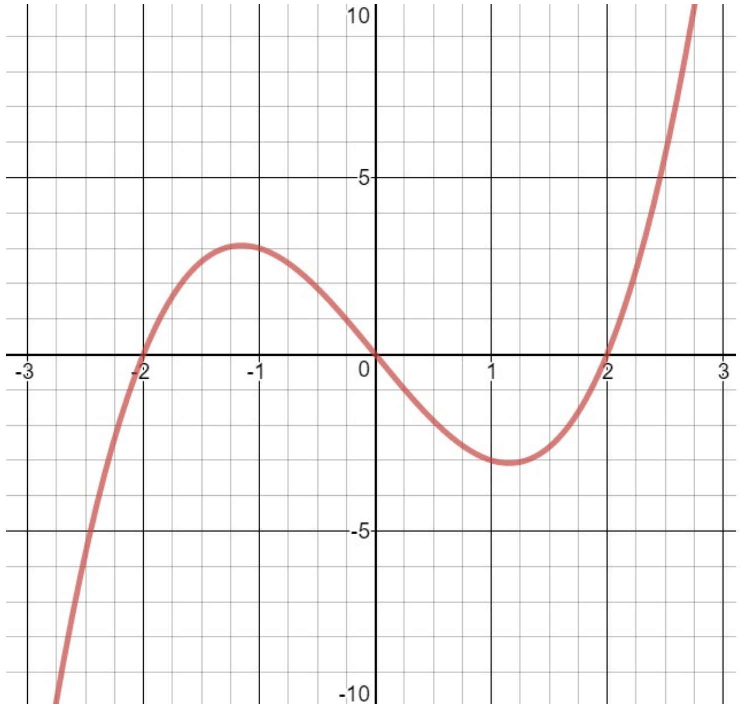


Clearly there is a pattern to these points. The derivative of the parabola is the linear function shown above

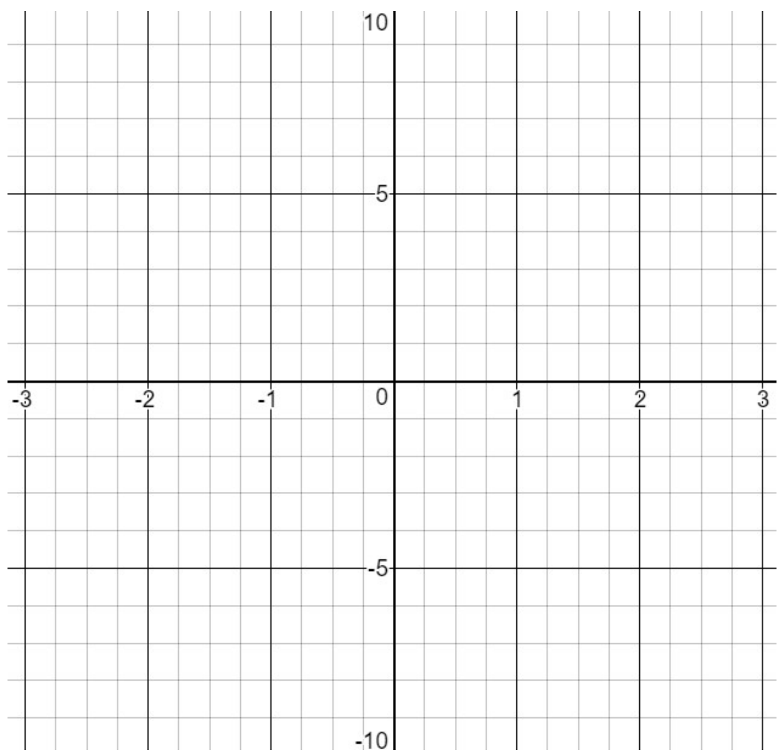
<u>Notes</u>

Practice

1. For the graph below, fill out the table with derivative values and graph them in the adjacent graph.



x	$f'(x)$



Question 2 – How do you calculate the derivative of a function from the definition?

Key Terms

Derivative

Difference Quotient

Function

Summary

The derivative function is defined by a difference quotient like the instantaneous rate or the derivative at a point:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Note that the difference quotient contains *x* instead of *a*. This is fortunate since it allows us to use the same algebraic techniques from Chapter 10 to evaluate the limits.

Notes

Guided Example

Use the definition of the derivative to find the derivative of

$$f(x) = 3x^2 + 4x + 1$$

Solution To evaluate the limit in the definition,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} ,$$

we need to evaluate $f(x + h)$ in the numerator. Replacing x with $x + h$ in the function gives

$$\begin{aligned} f(x+h) &= 3(x+h)^2 + 4(x+h) + 1 \\ &= 3x^2 + 6xh + 3h^2 + 4x + 4h + 1 \end{aligned}$$

Put this into the difference quotient along with $f(x)$:

$$\lim_{h \to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 + \cancel{4x} + 4h + \cancel{1} - \cancel{3x^2} - \cancel{4x} - \cancel{1}}{h}$$

$$\lim_{h \to 0} \frac{6xh + 3h^2 + 4h}{h}$$

$$\lim_{h \to 0} \frac{\cancel{h}(6x + 3h + 4)}{\cancel{h}}$$

$$\lim_{h \to 0} 6x + 3h + 4$$

$$3x + 4$$

So $f'(x) = 3x + 4$. Note that in the final limit, only h is approaching 0 so the derivative contains an x in it.

Practice

1. Use the definition of the derivative to find the derivative of

$$f(x) = -x^2 + 2x - 6$$

Guided Example	Practice
Use the definition of the derivative to find the derivative of $$g(x)=\frac{1}{x}$$ **Solution** Since the function is named $g(x)$, we need to adapt the definition to account for the different name: $$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$ As in the previous guided example, we need to find $g(x+h)$: $$g(x+h)=\frac{1}{x+h}$$ Put this into the difference quotient along with $g(x)$: $$\begin{aligned} g'(x)&=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}\\ &=\lim_{h\to 0}\frac{\frac{x}{x(x+h)}-\frac{x+h}{x(x+h)}}{h}\\ &=\lim_{h\to 0}\frac{\frac{\cancel{x}-\cancel{x}-h}{x(x+h)}}{h}\\ &=\lim_{h\to 0}\frac{-\cancel{h}}{x(x+h)}\cdot\frac{1}{\cancel{h}}\\ &=\lim_{h\to 0}\frac{-1}{x(x+h)}\\ &=\frac{-1}{x^2}\end{aligned}$$ The derivative is $g'(x)=\frac{-1}{x^2}$	2. Use the definition of the derivative to find the derivative of $$g(x)=\frac{1}{2x}$$ Solution

Question 3 – What are the derivatives of some basic functions (linear, polynomial, power, exponential, and logarithmic)?

Key Terms

Derivative	Polynomial function
Exponential function	Logarithm function

Summary

Calculating the derivative from the definition is tedious and time consuming. Luckily, formulas exist for calculating derivatives of basic functions like polynomial, exponential, and logarithmic functions.

$\frac{d}{dx}[c]=0$	$\frac{d}{dx}[ax+b]=a$	$\frac{d}{dx}\left[x^n\right]=nx^{n-1}$
$\frac{d}{dx}\left[a\,f(x)\right]=a\ f'(x)$	$\frac{d}{dx}\left[f(x)\pm g(x)\right]=f'(x)\pm g'(x)$	$\frac{d}{dx}\left[e^x\right]=e^x$
$\frac{d}{dx}\left[a^x\right]=(\ln a)a^x$	$\frac{d}{dx}\left[\log_a(x)\right]=\frac{1}{\ln(a)}\cdot\frac{1}{x}$	$\frac{d}{dx}\left[\ln(x)\right]=\frac{1}{x}$

Notes

Guided Example	Practice
Find the derivative of $f(x)=4x^2-5x+7$ **Solution** Utilize the rules and especially the power rule for derivatives to give $f'(x)=\frac{d}{dx}\left[4x^2\right]-\frac{d}{dx}[5x]+\frac{d}{dx}[7]$ $=4\frac{d}{dx}\left[x^2\right]-5\frac{d}{dx}[x]+\frac{d}{dx}[7]$ $=4\cdot 2x-5\cdot 1+0$ $=8x-5$	1. Find the derivative of $f(x)=5x^4-6x^3+1$

Guided Example	Practice
If $y=\frac{2}{x^2}+\pi^3$, find $\frac{dy}{dx}$. **Solution** Start by rewriting the fraction with negative exponents: $y=2x^{-2}+\pi^3$ Now take the derivative of each term: $\frac{dy}{dx}=\frac{d}{dx}\left[2x^{-2}\right]+\frac{d}{dx}\left[\pi^3\right]$ $=2\frac{d}{dx}\left[x^{-2}\right]+\frac{d}{dx}\left[\pi^3\right]$ $=2\cdot -2x^{-3}+0$ $=-4x^{-3}$ The second term is zero since it is constant with respect to the variable *x*. Only powers on the same variables change with the power rule.	2. Find $D_t\left[\frac{1}{2t}+e^2\right]$

Guided Example

Find the derivative of $g(t)=\sqrt{t}-t^{-1/2}$

Solution Rewrite the function so that all terms are written with powers. Using fractional powers in place of the root we get

$$g(t)=t^{1/2}-t^{-1/2}$$

Now utilize the power rule on each term to give

$$\begin{aligned}g'(t)&=\frac{d}{dt}\left[t^{1/2}\right]-\frac{d}{dt}\left[t^{-1/2}\right]\\&=\frac{1}{2}t^{-1/2}-\left(-\frac{1}{2}t^{-3/2}\right)\\&=\frac{1}{2}t^{-1/2}+\frac{1}{2}t^{-3/2}\end{aligned}$$

Practice

3. Find the derivative of $P(q)=\dfrac{\sqrt[4]{q}+q+1}{q}$

Guided Example

Find $D_x\left[3\cdot 2^x\right]$

Solution Apply the derivative rule for exponentials to

$$\begin{aligned}D_x\left[3\cdot 2^x\right]&=3\,D_x\left[2^x\right]\\&=3\cdot(\ln 2)\,2^x\end{aligned}$$

Practice

4. If $y=6\,e^x$, find $\dfrac{dy}{dx}$.

Guided Example	Practice
Find the derivative of $h(x) = \ln(x) + 5x^3 + 10$ **Solution** Take the derivative of each term: $h'(x) = \frac{d}{dx}[\ln(x)] + \frac{d}{dx}[5x^3] + \frac{d}{dx}[10]$ $= \frac{d}{dx}[\ln(x)] + 5\frac{d}{dx}[x^3] + \frac{d}{dx}[10]$ $= \frac{1}{x} + 5 \cdot 3x^2 + 0$ $= \frac{1}{x} + 15x^2$	5. Find the derivative of $h(x) = \log_3(x) - e^6 + 1$