Section 11.6 Derivatives of Products and Quotients

Question 1 – How do you find the derivative of a product of two functions?

Question 2 – How do you find the derivative of a quotient of two functions?

Question 3 - What is an average cost function?

Question 1 – How do you find the derivative of a product of two functions?

Key Terms

Product of two functions

Summary

Products are frequently encountered in business and finance. Products are where two functions are multiplied together. If the two functions multiplied together are called $u(x)$ and $v(x)$, we can take the derivative of the product using the Product Rule for Derivatives:

$$\frac{d}{dx}\left[u(x)v(x)\right] = v(x)u'(x) + u(x)v'(x)$$

This can be more easily memorized as

$$\frac{d}{dx}\left[uv\right] = vu' + uv'$$

Notes

Guided Example	Practice
Find the derivative of $y = xe^x$. **Solution** This function may be thought of as a product with $u = x$ and $v = e^x$. To apply the product rule, we need to take the derivative of each of these factors: $u = x \qquad u' = 1$ $v = e^x \qquad v' = e^x$ Now apply the product rule, $\frac{d}{dx}[uv] = vu' + uv'$ to give the derivative, $\frac{dy}{dx} = \underbrace{e^x}_{v} \cdot \underbrace{1}_{u'} + \underbrace{x}_{u} \cdot \underbrace{e^x}_{v'}$	1. Find the derivative of $y = (x^2 + 1)5^x$

Guided Example	Practice
Find $D_x\left[(x^2 + 7x - 5)\ln(x)\right]$. **Solution** Start by identifying the two factors and their derivatives: $u = x^2 + 7x - 5 \qquad u' = 2x + 7$ $v = \ln(x) \qquad v' = \frac{1}{x}$ Now put these pieces into the product rule to yield $\frac{dy}{dx} = \ln(x) \cdot \left(2x + 7\right) + \left(x^2 + 7x - 5\right) \cdot \frac{1}{x}$	2. Find $\frac{d}{dx}\left[(2x^3 - 5x + 1)\log(x)\right]$

Question 2 – How do you find the derivative of a quotient of two functions?

Key Terms

Quotient of two functions

Summary

When two functions are divided, the result is called a quotient. If the two pieces of the quotient are called $u(x)$ and $v(x)$, the we can take the derivative of the quotient with the Quotient Rule for derivatives:

$$\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right]=\frac{v(x)u'(x)-u(x)v'(x)}{\left(v(x)\right)^2}$$

This rule is more easily memorized in the form

$$\frac{d}{dx}\left[\frac{u}{v}\right]=\frac{vu'-uv'}{v^2}$$

Notes

Guided Example	Practice
Find $D_t\left[\frac{\log(t)}{t}\right]$ **Solution** Identify u and v so that we can apply the quotient rule, $$\frac{d}{dt}\left[\frac{u}{v}\right]=\frac{vu'-uv'}{v^2}$$ In this case, $u=\log(t) \qquad u'=\frac{1}{\ln(10)}\cdot\frac{1}{t}$ $v=t \qquad v'=1$ Now put these factors into the quotient rule, $$\frac{dy}{dt}=\frac{t\cdot\frac{1}{\ln(10)}\cdot\frac{1}{t}-\log(t)\cdot 1}{t^2}$$ $$=\frac{\frac{1}{\ln(10)}-\log(t)}{t^2}$$	1. Find the $\frac{dy}{dt}$ if $y=\frac{\ln(t)}{t}$.

<u>Guided Example</u>	<u>Practice</u>
If $f(x)=\dfrac{x+1}{x^2+4}$, find $f'(x)$. **Solution** For this function, $u=x+1 \qquad u'=1$ $v=x^2+4 \qquad v'=2x$ Put these factors into the quotient rule to give $f'(x)=\dfrac{\left(x^2+4\right)\cdot 1-\left(x+1\right)\cdot 2x}{\left(x^2+4\right)^2}$ $=\dfrac{x^2+4-2x^2-2x}{\left(x^2+4\right)^2}$ $=\dfrac{-x^2-2x+4}{\left(x^2+4\right)^2}$	2. If $g(t)=\dfrac{t^2-5t+10}{t+2}$, find $g'(t)$.

Question 3 – What is an average cost function?

Key Terms

Average Cost

Summary

If the total cost of producing Q items is $TC(Q)$, then the average cost function is found by dividing the total cost by the number of units Q,

$$\overline{TC}(Q) = \frac{TC(Q)}{Q}$$

The bar over the TC indicates average total cost. In many cases you will see the cost defined at x units as $C(x)$. In this case, you would write the average cost as

$$\overline{C}(x) = \frac{C(x)}{x}$$

The marginal average cost is simply the derivative of the average cost.

Notes

Guided Example	Practice
Suppose the total cost of producing Q items is given by $TC(Q) = 5Q + 6000$ thousand dollars.	1. Suppose the total cost of producing x items is given by $C(x) = 1.5x + 300$ thousand dollars.
a. Find the average cost function $\overline{TC}(Q)$.	a. Find the average cost function $\overline{C}(x)$.
Solution To find the average cost function, divide the total cost by the quantity, $$\overline{TC}(Q) = \frac{5Q + 6000}{Q}$$ Since the total cost is in thousands of dollars, the average cost is in thousands of dollars per item.	
b. Find and interpret $\overline{TC}(200)$.	b. Find and interpret $\overline{C}(100)$.
Solution Substitute $Q = 200$ into the average cost to yield $$\overline{TC}(200) = \frac{5(200) + 6000}{200} = 35$$ The average cost is 35 thousand dollars per item which means that each item costs $35,000 to make.	
c. Find the marginal average cost function.	c. Find the marginal average cost function.
Solution To find the marginal average cost function, we need to take the derivative of the average cost function $\overline{TC}(Q) = \frac{5Q + 6000}{Q}$. Since the function is a quotient, let $$u = 5Q + 6000 \qquad u' = 5$$ $$v = Q \qquad v' = 1$$ and put these factors into the quotient rule,	

$$\overline{TC}'(Q) = \frac{Q \cdot 5 - (5Q + 6000) \cdot 1}{Q^2} = \frac{-6000}{Q^2}$$ The units on this function are thousands of dollars per unit per unit.	
d. Find and interpret the marginal average cost function when 200 items are made. **Solution** Substitute $Q = 200$ into the function from part c. This gives $$\overline{TC}'(200) = \frac{-6000}{200^2} = -0.15$$ This means that when production in increased by 1 item, the average cost will decrease by 0.15 thousand dollars per unit.	d. Find and interpret the marginal average cost function when 100 items are made.