Section 11.7 The Chain Rule

Question 1 – What is a composition of two functions?

Question 2 – How do you write a function as a composition of two functions?

Question 3 - How do you apply the chain rule to take a derivative?

Question 4 - How do you combine derivative rules to take more complicated derivatives?

Question 1 – What is a composition of two functions?

Key Terms

Composition

Summary

When we compose two functions, we substitute one function in place of the variable in the other function. If the two functions are called f (x) and $g(x)$, then the composition is named $f \circ g$ or g $g \circ f$. The symbol ° indicates composition and would be read as "circle" or "composed with". The difference between $f \circ g$ and $g \circ f$ has to do with which function is substituted into which function. By definition, $f \circ g$ is obtained by substituting $g(x)$ into $f(x)$, $f\left(g(x)\right)$. The composition $g \circ f$ is obtained by substituting $f(x)$ into $g(x)$, $g\left(f(x)\right)$.

Notes

Guided Example	Practice
Suppose $f(x)=\frac{1}{2}x+7$ and $g(x)=5x+6$. a. Find $f(g(x))$. **Solution** Start by replacing $g(x)$ with its formula. Then place that function into $f(x)$: $f\left(g(x)\right)=f\left(5x+6\right)$ $=\frac{1}{2}\left(5x+6\right)+7$ b. Find $g(f(x))$. **Solution** Start by replacing $f(x)$ with its formula. Then place that function into $g(x)$: $g\left(f(x)\right)=g\left(\frac{1}{2}x+7\right)$ $=5\left(\frac{1}{2}x+7\right)+6$	1. Suppose $f(x)=\frac{3}{4}x+1$ and $g(x)=2x-4$. a. Find $f(g(x))$. b. Find $g(f(x))$.

Guided Example	Practice
Suppose $f(x)=\dfrac{1}{x}$ and $g(x)=2x^2-x+2$. a. Find $f(g(x))$. **Solution** Start by replacing $g(x)$ with its formula. Then place that function into $f(x)$: $f\left(g(x)\right)=f\left(2x^2-x+2\right)$ $=\dfrac{1}{2x^2-x+2}$ b. Find $g(f(x))$. **Solution** Start by replacing $f(x)$ with its formula. Then place that function into $g(x)$: $g\left(f(x)\right)=g\left(\frac{1}{x}\right)$ $=2\left(\frac{1}{x}\right)^2-\left(\frac{1}{x}\right)+2$	2. Suppose $f(x)=\dfrac{6}{x}$ and $g(x)=x^3-x^2-5$. a. Find $f(g(x))$. b. Find $g(f(x))$.

Guided Example	Practice
Suppose $f(x)=\sqrt{x-2}$ and $g(x)=\frac{2}{x}$. a. Find $f(g(x))$. **Solution** Start by replacing $g(x)$ with its formula. Then place that function into $f(x)$: $f(g(x)) = f\left(\frac{2}{x}\right) = \sqrt{\frac{2}{x}-2}$ b. Find $g(f(x))$. **Solution** Start by replacing $f(x)$ with its formula. Then place that function into $g(x)$: $g(f(x)) = g\left(\sqrt{x-2}\right) = \frac{2}{\sqrt{x-2}}$	3. Suppose $f(x)=\sqrt[3]{x+1}$ and $g(x)=\frac{1}{x}$. a. Find $f(g(x))$. b. Find $g(f(x))$.

Guided Example	Practice
Suppose the demand for a certain product is given by $D(p)=\frac{1}{p+1}$, where p is in dollars. If the price, in terms of cost c, is expressed as $p(c)=5c-10$, find the demand function in terms of cost. **Solution** To find the demand function as a function of cost, we need to compose the demand function with the price function. This allows us to combine the price function (which takes in cost and outputs price) with the demand function (which takes in price and outputs demand). $D(p(c)) = D(5c-10) = \frac{1}{5c-10+1}$	4. Suppose the demand for a certain product is given by $D(p)=\frac{-p^2}{50}+125$, where p is in dollars. If the price, in terms of cost c, is expressed as $p(c)=2c+15$, find the demand function in terms of cost.

Question 2 – How do you write a function as a composition of two functions?

Key Terms

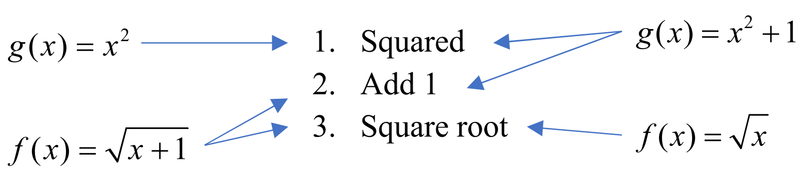
Decomposition

Summary

When function is written as a composition of two functions, we say the function has been decomposed. This process is the reverse of composing two functions. Suppose we have a function $h(x)=\sqrt{x^2+1}$. To see how we can decompose this into $f(x)$ and $g(x)$ so that $f\left(g(x)\right)$, let's examine what the function does. If we put an input into $h(x)$, the input is

1. Squared
2. Add 1
3. Square root

When this function is decomposed, one of the functions needs to perform two of these steps and the other need to perform the other step.



Either decomposition will lead to the same function when composed as $f\left(g(x)\right)$.

Notes

Guided Example	Practice
Write the function below as a decomposition of two functions in the form $f(g(x))$. $y=\left(x^2+1\right)^6$ **Solution** If we list out the steps that this function performs on an input, we get 1. Square 2. Add 1 3. Raise to the 6th power We can decompose these steps as $g(x)=x^2$ and $f(x)=\left(x+1\right)^6$ or as $g(x)=x^2+1$ and $f(x)=x^6$. Either decomposition results in the same function when composed.	1. Write the function below as a decomposition of two functions in the form $f(g(x))$. $y=\left(x^3+x\right)^5$

Guided Example	Practice
Write the function below as a decomposition of two functions in the form $f(g(x))$. $y=\sqrt{x^2-2x+5}$ **Solution** This function is more complicated since there is more than one place to put the input. However, both of those inputs are in a polynomial. So, let $g(x)=x^2-2x+5$. To get the proper composition, the other function must be $f(x)=\sqrt{x}$.	2. Write the function below as a decomposition of two functions in the form $f(g(x))$. $y=\sqrt[3]{2x^2+x-15}$

Question 3 – How do you apply the chain rule to take a derivative?

<u>Key Terms</u>

Chain rule Differentiable

<u>Summary</u>

If a function may be written as a composition $f\left(g(x)\right)$, its derivative is computed using the Chain Rule. In the Chain Rule, we decompose the function into two differentiable functions $f(x)$ and $g(x)$. Then the derivative of the composition is

$$\frac{d}{dx}\left[f\left(g(x)\right)\right]=f'\left(g(x)\right)g'(x)$$

Notice that the chain rule results in a product. In the first factor, $g(x)$ is substituted into the derivative $f'(x)$. The second factor is the derivative $g'(x)$.

<u>Notes</u>

Guided Example	Practice
Find the derivative of the function, $y=\left(x^3-2x+6\right)^5$ **Solution** Start by decomposing the function into $f(x)=x^5 \qquad g(x)=x^3-2x+6$ To apply the chain rule, we need the derivatives of each of these pieces: $f'(x)=5x^4 \qquad g'(x)=3x^2-2$ Now put these pieces into the Chain Rule, $\frac{dy}{dx}=\underbrace{5\left(x^3-2x+6\right)^4}_{f'(g(x))}\underbrace{\left(3x^2-2\right)}_{g'(x)}$	1. Find the derivative of the function, $y=\left(x^2+1\right)^{10}$

Guided Example	Practice
Find the derivative of the function, $y=-4\left(2x^3+3x^2\right)^{-4}$ **Solution** Start by decomposing the function into $f(x)=-4x^{-4} \qquad g(x)=2x^3+3x^2$ To apply the chain rule, we need the derivatives of each of these pieces: $f'(x)=16x^{-5} \qquad g'(x)=6x^2+6x$ Now put these pieces into the Chain Rule, $\frac{dy}{dx}=\underbrace{16\left(2x^3+3x^2\right)^{-5}}_{f'(g(x))}\underbrace{\left(6x^2+6x\right)}_{g'(x)}$	2. Find the derivative of the function, $y=\left(x^2+10\right)^{-5}$

Guided Example	Practice
Find the derivative of the function, $$y=10(2t+7)^{2/3}$$ **Solution** Start by decomposing the function into $$f(t)=10t^{2/3} \qquad g(t)=2t-7$$ To apply the chain rule, we need the derivatives of each of these pieces: $$f'(t)=\tfrac{20}{3}t^{-1/3} \qquad g'(t)=2$$ Now put these pieces into the Chain Rule, $$\frac{dy}{dt}=\underbrace{\tfrac{20}{3}(2t-7)^{-1/3}}_{f'(g(t))}\underbrace{(2)}_{g'(t)}$$	3. Find the derivative of the function, $$y=6(3t-5)^{1/3}$$

Guided Example	Practice
The cost for producing x units of a product is $$C(x)=\sqrt{2x^2+250}$$ where C is the cost in dollars. Find the marginal cost of producing 35 units and interpret the answer. **Solution** To find the marginal cost, we need to take the derivative of the cost function. This requires us to use the Chain Rule with $$f(x)=\sqrt{x} \qquad g(x)=2x^2+250$$ To apply the chain rule, we need the derivatives of each of these pieces: $$f'(x)=\tfrac{1}{2}x^{-1/2} \qquad g'(x)=4x$$ Now put these pieces into the Chain Rule,	4. The cost for producing x units of a product is $$C(x)=\sqrt[3]{5x+10}$$ where C is the cost in dollars. Find the marginal cost of producing 10 units and interpret the answer.

$$C'(x) = \underbrace{\tfrac{1}{2}\left(2x^2+250\right)^{-1/2}}_{f'(g(x))}\underbrace{\left(4x\right)}_{g'(x)}$$ $$= \frac{4x}{2\sqrt{2x^2+250}}$$ $$= \frac{2x}{\sqrt{2x^2+250}}$$ To find the marginal cost at 35 units, put $x = 35$ into the marginal cost function, $$C'(35) = \frac{2(35)}{\sqrt{2(35)^2+250}} \approx 1.35$$ A marginal cost of 1.35 dollars per unit means that producing the 36^{th} unit will increase cost by \$1.35.	

Question 3 – How do you combine derivative rules to take more complicated derivatives?

Key Terms

Product Rule

Quotient Rule

Chain Rule

Summary

In more complicated derivatives, the Chain Rule may be combined with the Product Rule,

$$\frac{d}{dx}[uv] = vu' + uv'$$

or the Quotient Rule,

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$$

Separating the rules from the Chain Rule help you to process the rules individually so they are not confused.

Notes

Guided Example

Find the derivative of the function,

$$y=\left(x^2+4x\right)\left(5x+6\right)^4$$

Solution Start by noticing that this function is a product with a composition in the second piece of the product. This tells us that we will start with the Product Rule and then apply the Chain Rule when we need the derivative of the second factor.

Starting with the Product Rule, we get

$$\frac{dy}{dx}=\left(5x+6\right)^4\left(2x+4\right)+\left(x^2+4x\right)\cdot\frac{d}{dx}\left[\left(5x+6\right)^4\right]$$

The Product Rule has been written out. Notice that we left one of the derivatives in symbolic form. Next, we will use the Chain Rule to carry out that derivative:

$$\frac{d}{dx}\left[\left(5x+6\right)^4\right]=4\left(5x+6\right)^3\left(5\right)$$

Placing this in our Product Rule completes the derivative,

$$\frac{dy}{dx}=\left(5x+6\right)^4\left(2x+4\right)+\left(x^2+4x\right)\cdot 4\left(5x+6\right)^3\left(5\right)$$

Practice

1. Find the derivative of the function,

$$y=x\left(2x^2-x\right)^3$$

Guided Example

Find the derivative of the function,

$$y = x\, e^{x^2+1}$$

Solution This is a Product Rule with a Chain Rule inside of one of the factors. Using the Product Rule, we get

$$\frac{dy}{dx} = e^{x^2+1}(1) + x \cdot \frac{d}{dx}\left[e^{x^2+1}\right]$$

Applying the Chain Rule to the derivative that still needs to be completed, we get

$$\frac{d}{dx}\left[e^{x^2+1}\right] = e^{x^2+1}(2x)$$

Combining this with the Product Rule above gives

$$\frac{dy}{dx} = e^{x^2+1}(1) + x \cdot e^{x^2+1}(2x)$$

Practice

2. Find the derivative of the function,

$$y = x^3 \ln(2x+1)$$

Guided Example

Find the derivative of the function,

$$y = \frac{2x+1}{\ln(4x+7)}$$

Solution This derivative requires us to use the Quotient Rule followed by the Chain Rule. Starting with the Quotient Rule, we get

$$\frac{dy}{dx} = \frac{\ln(4x+7)(2) - (2x+1)\frac{d}{dx}\left[\ln(4x+7)\right]}{\left(\ln(4x+7)\right)^2}$$

Now apply the Chain Rule to finish the remaining derivative:

$$\frac{d}{dx}\left[\ln(4x+7)\right] = \frac{1}{4x+7} \cdot 4$$

Put this derivative into the Quotient Rule to give

$$\frac{dy}{dx} = \frac{\ln(4x+7)(2) - (2x+1)\left(\frac{4}{4x+7}\right)}{\left(\ln(4x+7)\right)^2}$$

Practice

3. Find the derivative of the function,

$$y = \frac{\ln(2x+1)}{4x+7}$$