Section 13.1 Antiderivatives

Question 1 – What is an antiderivative?

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Question 4 - How do we find the value of the arbitrary constant?

Question 1 – What is an antiderivative?

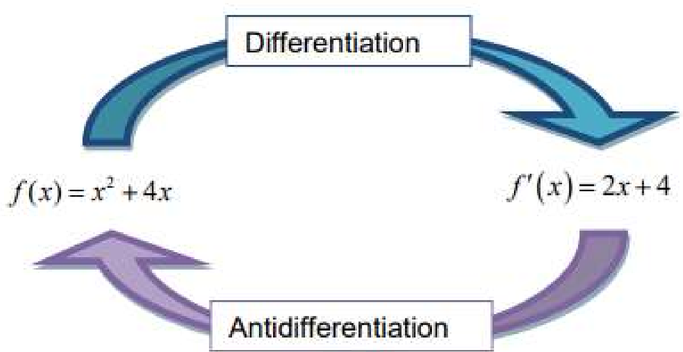
Key Terms

Antiderivative   Antidifferentiation

Summary

In earlier chapters, we learned how to take the derivative of a function. We had many rules (especially the Power Rule for derivatives) that allowed us to take a function like $f(x) = x^2 + 4x$ and to find its derivative $f'(x) = 2x + 4$. This process, called differentiation, helps us to compute derivatives of functions.

In this question, we look at the reverse process called antidifferentiation. Now we start from the derivative and then compute the function the derivative came from. This function is called the antiderivative.



For the functions pictured above, we would say the antiderivative of $f'(x) = 2x + 4$ is $f(x) = x^2 + 4x$.

We can often deduce the antiderivative from our knowledge of derivative rules. In the case above, we know the derivative reduces the power by one. So the antiderivative must do the opposite or add one to the power. Knowledge of the derivative rules can often get us very close to the antiderivative. We can always fine tune our educated guesses by taking the derivative of the antiderivative.

When taking derivatives, we used the symbol $\frac{d}{dx}[\ ]$ to indicate that we want to take the derivative of the expression in brackets. For the example above, we would write

$$\frac{d}{dx}\left[x^2+4x\right]=2x+4$$

For antiderivatives, we use the symbol $\int(\ )\,dx$ to indicate the antiderivative of the function in parentheses. So, we could write

$$\int(2x+4)\,dx=x^2+4x+C$$

To indicate that the antiderivative of $2x+4$ is x^2+4x . An arbitrary constant *C* is added to the antiderivative because there are many antiderivatives of $2x+4$. Each antiderivative has a different constant symbolized by the *C*.

<u>Notes</u>

<table>
<tr><th>Guided Example</th><th>Practice</th></tr>
<tr><td>Find the antiderivative of $f'(x) = 3x^2 + 2x + 5$.

Solution Since you are given the derivative, you need to reverse the rules for derivatives. In this case, you are reversing the Power Rule
$$\frac{d}{dx}\left[x^n\right] = n\,x^{n-1}$$
Examine each term carefully and ask yourself, “What would you take the derivative of to get each term in $3x^2 + 2x + 5$?”

Based on your experience with derivatives, you would probably realize that the derivative of x^3 is $3x^2$ and that the derivative of x^2 is $2x$. But what about the last term?

Since the derivative of $5x$ is 5, we can deduce the antiderivative to be
$$f(x) = x^3 + x^2 + 5x + C$$
where C is some unknown constant.</td><td>1. Find the antiderivative of
$$f'(x) = 5x^4 + 4x^3 - 2$$</td></tr>
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<table>
<tr><th>Guided Example</th><th>Practice</th></tr>
<tr><td>Find each integral

a. $\int 5x^4 z\,dz$

Solution The variable in this antiderivative is z. Because of this, we can ignore $5x^4$ and focus on the antiderivative of z. Rephrasing this, ask yourself, “What would you take the derivative of to get z?” Certainly, the power must be one higher so would z^2 work?

Close, but the derivative of z^2 is $2z$. So, try $\frac{1}{2}z^2$. Since $\frac{d}{dz}\left[\frac{1}{2}z^2\right] = z$, the antiderivative is
$$\int 5x^4 z\,dz = 5x^4 \cdot \frac{1}{2}z^2 + C$$</td><td>2. Find each integral

a. $\int 10p^9 q\,dp$.</td></tr>
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b. $\int 5x^4 z\, dx$ **Solution** In this antiderivative the variable is x. We can consider the z a constant and simply look for the antiderivative of $5x^4$. This antiderivative is x^5 . Putting this together, we get $$\int 5x^4 z\, dx = x^5 \cdot z + C$$	b. $\int 10 p^9 q\, dq$.

<u>Notes</u>

Question 2 – What are the antiderivatives of some basic functions?

Key Terms

Antiderivative

Summary

The rules for taking derivatives can be reversed to obtain the antiderivative rules. From Section 11.4, here are the derivative rules.

$\frac{d}{dx}\left[ax+b\right]=a$	$\frac{d}{dx}\left[x^n\right]=nx^{n-1}$	$\frac{d}{dx}\left[e^x\right]=e^x$
$\frac{d}{dx}\left[a^x\right]=(\ln a)a^x$	$\frac{d}{dx}\left[\log_a(x)\right]=\frac{1}{\ln(a)}\cdot\frac{1}{x}$	$\frac{d}{dx}\left[\ln(x)\right]=\frac{1}{x}$

The corresponding antiderivative rules are

$\int a\,dx = ax + C$	$\int x^n\,dx = \frac{x^{n+1}}{n+1}+C$ for $n \neq -1$	$\int e^x\,dx = e^x + C$
$\int a^x\,dx = \frac{a^x}{\ln(a)}+C$	$\int \frac{1}{x}\,dx = \ln(x)+C$ for $x>0$	

Notes

Guided Example	Practice
Evaluate $\int x^{12}\ dx$ **Solution** Apply the Power Rule for Antiderivatives, $$\int x^n\ dx = \frac{x^{n+1}}{n+1} + C$$ with $n = 12$. This gives $$\int x^{12}\ dx = \frac{x^{13}}{13} + C$$	1. Evaluate $\int x^{20}\ dx$

Guided Example	Practice
Evaluate $\int \sqrt[3]{z}\ dz$ **Solution** To apply the Power Rule for Antiderivatives, rewrite the root $\sqrt[3]{z} = z^{1/3}$. This will give us, $$\int z^{1/3}\ dz = \frac{z^{1/3+1}}{1/3+1} + C = \frac{z^{4/3}}{4/3} + C = \frac{3}{4} z^{4/3} + C$$	2. Evaluate $\int \sqrt[4]{x}\ dx$

Guided Example	Practice
Evaluate $\int \frac{1}{t^5}\,dt$ **Solution** To apply the Power Rule for Antiderivatives, rewrite $\frac{1}{t^2} = t^{-2}$. This will give us, $$\int t^{-2}\,dt = \frac{t^{-2+1}}{-2+1} + C = \frac{t^{-1}}{-1} + C = -\frac{1}{t} + C$$	3. Evaluate $\int \frac{1}{u^3}\,du$

Guided Example	Practice
Evaluate $\int \frac{1}{z}\,dz$ **Solution** Like the last guided example, we might try to write $\frac{1}{z} = z^{-1}$. However, the Power Rule for Antiderivatives does not apply when n = -1. Instead we apply a different rule, $$\int \frac{1}{x}\,dx = \ln(x) + C \quad \text{for } x > 0$$ but with a different variable. Using z instead of x gives $$\int \frac{1}{z}\,dz = \ln(z) + C \quad \text{for } z > 0$$	4. Evaluate $\int 3^x\,dx$

Question 3 – How do we find the antiderivative of functions that are combinations of basic functions?

Key Terms

Antiderivative

Summary

Constants are all but ignored by the derivative. A similar property exists for antiderivatives that says for any real number constant *a*,

$$\int a\ f(x)\ dx = a\int f(x)\ dx$$

In effect, we can ignore the constant when taking an antiderivative and tack it on at the end.

Another property exists for breaking more complicated antiderivatives into smaller pieces. If you have a sum or difference of functions,

$$\int \left(f(x) \pm g(x) \right) dx = \int f(x) dx \pm \int g(x) dx$$

This means that we can break up sums and differences and compute the antiderivatives of the resulting pieces.

Notes

Guided Example

Evaluate $\int \frac{1}{2x}\,dx$

Solution The constant may be moved outside of the antiderivative and then we may take the antiderivative:

$$\int \frac{1}{2x}\,dx = \frac{1}{2}\int \frac{1}{x}dx$$

$$= \frac{1}{2}\ln(x) + C \text{ for } x > 0$$

Practice

1. Evaluate $\int 5P^2\,dP$

Guided Example

Evaluate $\int \left(2x^4 - 5x^2 + 2x + 7\right)dx$

Solution Our rules for constants and sums or differences to break the polynomial into smaller pieces. The pieces can be evaluated with the Power Rule for Antiderivatives and the Constant Rule for Antiderivativess:

$$\int \left(2x^4 - 5x^2 + 2x + 7\right)dx = 2\int x^4 dx - 5\int x^2 dx + 2\int x\,dx + \int 7dx$$

$$= 2\frac{x^5}{5} - 5\frac{x^3}{3} + 2\frac{x^2}{2} + 7x + C$$

$$= \frac{2}{5}x^5 - \frac{5}{3}x^3 + x^2 + 7x + C$$

Practice

2. Evaluate $\int \left(6x^3 - 7x^2 + 10x - 5\right)dx$

Guided Example

Evaluate $\int 10z\left(z^4 - 7z + 2\right)dz$

Solution Multiply the factors out and apply the Power Rule for Antiderivatives:

$$\begin{aligned}\int 10z\left(z^4 - 7z + 2\right)dz &= \int\left(10z^5 - 70z^2 + 20z\right)dz \\ &= 10\int z^5 dz - 70\int z^2 dz + 20\int z\,dz \\ &= 10\frac{z^6}{6} - 70\frac{z^3}{3} + 20\frac{z^2}{2} + C \\ &= \frac{5}{3}z^6 - \frac{70}{3}z^3 + 10z^2 + C\end{aligned}$$

Practice

3. Evaluate $\int\left(x^2 - 1\right)\left(x + 1\right)dx$

Guided Example	Practice
Evaluate $\int \frac{x^3 - 2x}{x}\,dx$ **Solution** Divide the x in the denominator into each term in the numerator and then apply antiderivative rules: $\int \frac{x^3 - 2x}{x}\,dx = \int \left(\frac{x^3}{x} - \frac{2x}{x}\right)dx$ $= \int \left(x^2 - 2\right)dx$ $= \int x^2 dx - \int 2\,dx$ $= \frac{x^3}{3} - 2x + C$	1. Evaluate $\int \frac{\sqrt{x} - 1}{x}\,dx$

Question 4 – How do we find the value of the arbitrary constant?

Key Terms

Antiderivative

Summary

All of our antiderivative rules for basic functions contain an arbitrary constant C. This is because when we take the derivative of constants, the result is zero. For instance,

$$\frac{d}{dx}\left[x^2+4x+1\right]=2x+4$$

$$\frac{d}{dx}\left[x^2+4x+10\right]=2x+4$$

$$\frac{d}{dx}\left[x^2+4x+99\right]=2x+4$$

Each derivative results in the same function. If we reverse the process, we can get many different antiderivatives. These antiderivatives differ by a constant. We would indicate this by writing

$$\int(2x+4)\,dx=x^2+4x+C$$

To single out a particular antiderivative, you need some information about it. Suppose I know that the antiderivative must pass through (2, 22). This means that when I put $x = 2$ into the antiderivative, the resulting y value should be $y = 22$:

$$2^2+4(2)+C=22$$

If we solve this for C we have the value of the constant:

$$12+C=22$$

$$C=10$$

So $f(x)=x^2+4x+10$ is the antiderivative of $f'(x)=2x+4$ that passes through (2, 22).

Notes

Guided Example	Practice
The derivative of a function $f(x)$ is $f'(x)=5x-12$ Find the function $f(x)$ that passes through (2, 7). **Solution** The antiderivative of this function is $f(x)=\frac{5}{2}x^2-12x+C$ To make sure the function passes through (2, 7), substitute $x = 2$ into $f(x)$ and set the result equal to 7: $f(2)=\frac{5}{2}(2)^2-12(2)+C=7$ Solve this equation for C to give, $\frac{5}{2}(2)^2-12(2)+C=7$ $10-24+C=7$ $-14+C=7$ $C=21$ So, the function f (x) that passes through (2, 7) is $f(x)=\frac{5}{2}x^2-12x+21$	1. The derivative of a function $f(x)$ is $f'(x)=2x-6$ Find the function $f(x)$ that passes through (5, 10).

Guided Example	Practice
Find the cost function if the marginal cost is given $$C'(x) = x^{1.1} + 5$$ and 5 units costs \$359.69. **Solution** To find the cost function, we need to compute the antiderivative of $C'(x)$, $$\begin{aligned} C(x) &= \int \left(x^{1.1} + 5\right) dx \\ &= \int x^{1.1}\, dx + \int 5\, dx \\ &= \frac{x^{2.1}}{2.1} + 5x + C \end{aligned}$$ To make sure 5 units costs \$359.69, substitute x = 5 into the cost function and set the resulting expression equal to 359.69: $$C(5) = \frac{(5)^{2.1}}{2.1} + 5(5) + C = 359.69$$ Now solve the equation for C: $$\begin{aligned} \frac{(5)^{2.1}}{2.1} + 5(5) + C &= 359.69 \\ 13.984 + 25 + C &\approx 359.69 \\ 38.984 + C &\approx 359.69 \\ C &\approx 320.706 \end{aligned}$$ Put this constant into the antiderivative to yield $$C(x) = \frac{x^{2.1}}{2.1} + 5x + 320.706$$ Where the constant has been rounded to three decimal places.	2. Find the cost function if the marginal cost is given $$C'(x) = 2x + 25$$ and 10 units costs \$367.20.

Guided Example	Practice
Find the demand function $p(x)$ for the marginal revenue function $$R'(x) = 0.3x^2 - 0.4x + 112$$ Assume that if no items are sold, the revenue is 0. **Solution** Start by finding the revenue function from the marginal revenue function. Take the antiderivative of the marginal revenue to get $$\begin{aligned} R(x) &= \int \left(0.3x^2 - 0.4x + 112\right) dx \\ &= 0.1x^3 - 0.2x^2 + 112x + C \end{aligned}$$ We can find the value of C from the fact that revenue is zero when nothing is sold. This tells us that $R(0) = 0$. Put this information into the revenue function and solve for C: $$R(0) = 0.1(0)^3 - 0.2(0)^2 + 112(0) + C = 0$$ This leads to $C = 0$ and the resulting revenue function is $$R(x) = 0.1x^3 - 0.2x^2 + 112x$$ The demand function $p(x)$ is related to revenue $R(x)$ by the equation $$R(x) = x\ p(x)$$ Solving for p(x) gives $$p(x) = \frac{R(x)}{x}$$ Substitute the revenue into this equation to get $$\begin{aligned} p(x) &= \frac{0.1x^3 - 0.2x^2 + 112x}{x} \\ &= \frac{0.1x^3}{x} - \frac{0.2x^2}{x} + \frac{112x}{x} \\ &= 0.1x^2 - 0.2x + 112 \end{aligned}$$	3. Find the demand function $p(x)$ for the marginal revenue function $$R'(x) = 0.6x^2 - 0.8x + 127$$ Assume that if no items are sold, the revenue is 0.