Section 13.2 Approximating Area

Question 1 – Why is area important?

Question 2 – How is the area under a function approximated?

Question 1 – Why is area important?

Key Terms

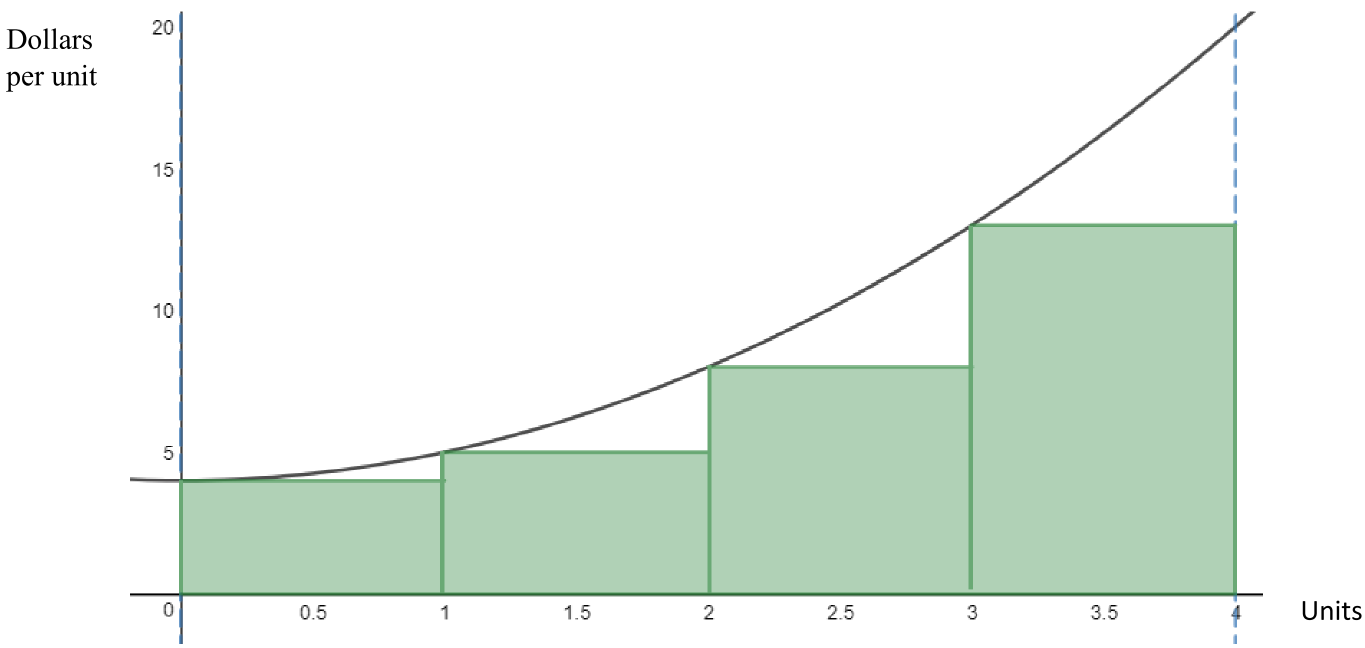
Area | Right Hand Sum

Left Hand Sum

Summary

When a function is a rate such as marginal profit, the area under the function corresponds to a change in the profit function. We can approximate this change with estimates constructed from rectangles whose heights correspond to the rate.

Below is a marginal profit function $P'(x) = x^2 + 4$.



If the marginal profit is in dollars per unit, the area of each rectangle has units of

$$\left(\frac{\text{dollars}}{\cancel{\text{unit}}}\right)\left(\cancel{\text{units}}\right) = \text{dollars}$$

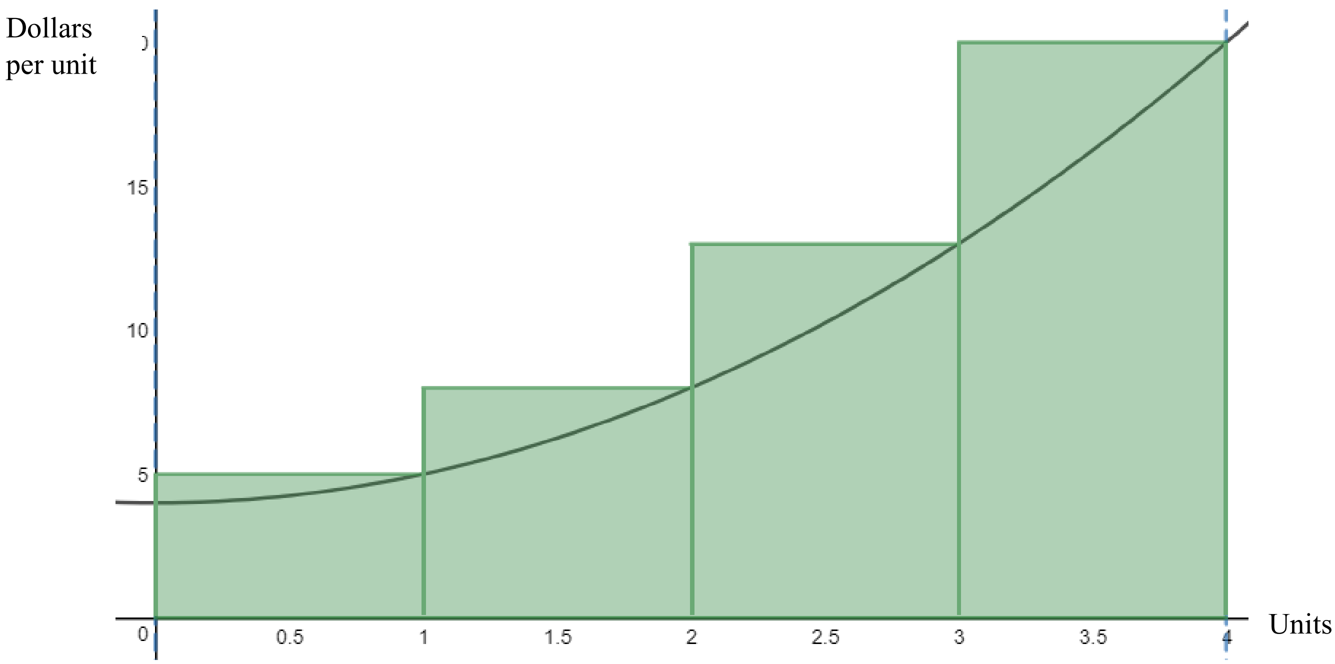
These are the units of profit. We can estimate the change in profit from x = 0 to x = 4 by summing the area of the rectangles:

$$\text{Estimate} = \left(4\frac{\text{dollars}}{\text{units}}\right)(1\text{ unit}) + \left(5\frac{\text{dollars}}{\text{units}}\right)(1\text{ unit}) + \left(8\frac{\text{dollars}}{\text{units}}\right)(1\text{ unit}) + \left(13\frac{\text{dollars}}{\text{units}}\right)(1\text{ unit})$$

$$= 30\text{ dollars}$$

Since each rectangle touches the graph on the left-hand side of the rectangle, this is called a left-hand estimate.

If the rectangles touch on the right-hand side of the rectangle, the estimate is called a right-hand estimate.



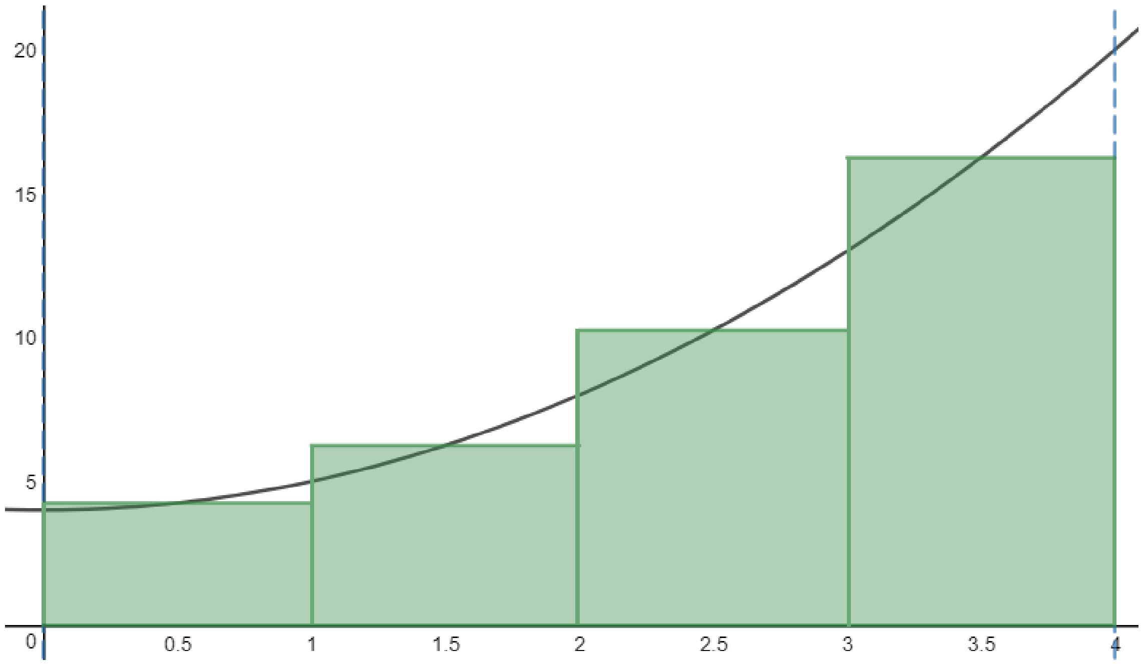
In this case the sum of the areas of the rectangles is

$$\text{Estimate} = \left(5\frac{\text{dollars}}{\text{units}}\right)(1\text{ unit}) + \left(8\frac{\text{dollars}}{\text{units}}\right)(1\text{ unit}) + \left(13\frac{\text{dollars}}{\text{units}}\right)(1\text{ unit}) + \left(20\frac{\text{dollars}}{\text{units}}\right)(1\text{ unit})$$

$$= 46\text{ dollars}$$

This estimate is an overestimate of the actual change in profit.

If the heights of the rectangles are found using the midpoint of each rectangle, the estimate is called a midpoint estimate.



The sum of the areas of these rectangles is

$$\text{Estimate} = \left(4.25\frac{\text{dollars}}{\text{units}}\right)(1\text{ unit}) + \left(6.25\frac{\text{dollars}}{\text{units}}\right)(1\text{ unit}) + \left(10.25\frac{\text{dollars}}{\text{units}}\right)(1\text{ unit}) + \left(16.25\frac{\text{dollars}}{\text{units}}\right)(1\text{ unit})$$

$$= 37\text{ dollars}$$

<u>Notes</u>

Guided Example

Use the table to find a lower and upper estimate of the change in profit when production is increased from 50 to 90 units.

Units	50	60	70	80	90
Rate of Change of Profit (Dollars per unit)	100	102	105	109	114

Solution Let's look at the interval from 50 to 60 units. The change in profit over this interval may be underestimated as

$$\text{Under} = \left(100\frac{\text{dollars}}{\text{unit}}\right)(10 \text{ units}) = 1000 \text{ dollars}$$

The overestimate is

$$\text{Over} = \left(102\frac{\text{dollars}}{\text{unit}}\right)(10 \text{ units}) = 1020 \text{ dollars}$$

The change in profit from 60 to 70 units may be underestimated as

$$\text{Under} = \left(102\frac{\text{dollars}}{\text{unit}}\right)(10 \text{ units}) = 1020 \text{ dollars}$$

Or overestimated as

$$\text{Over} = \left(105\frac{\text{dollars}}{\text{unit}}\right)(10 \text{ units}) = 1050 \text{ dollars}$$

Continue this process with similar products from 70 to 80 units as well as 80 to 90 units. The sum of the underestimates is

$$\text{Under} = (100)(10) + (102)(10) + (105)(10) + (109)(10) = 4160$$

This tells us that an underestimate of the change in profit from 50 to 90 units is $4160.

Summing the overestimates gives

$$\text{Under} = (102)(10) + (105)(10) + (109)(10) + (114)(10) = 4300$$

This tells us that an overestimate of the change in profit from 50 to 90 units is $4300.

Practice

1. Use the table to find a lower and upper estimate of the change in profit when production is increased from 10 to 50 units.

Units	10	20	30	40	50
Rate of Change of Profit (dollars per unit)	30	29	27	24	20

Question 2 – How is the area under a function approximated?

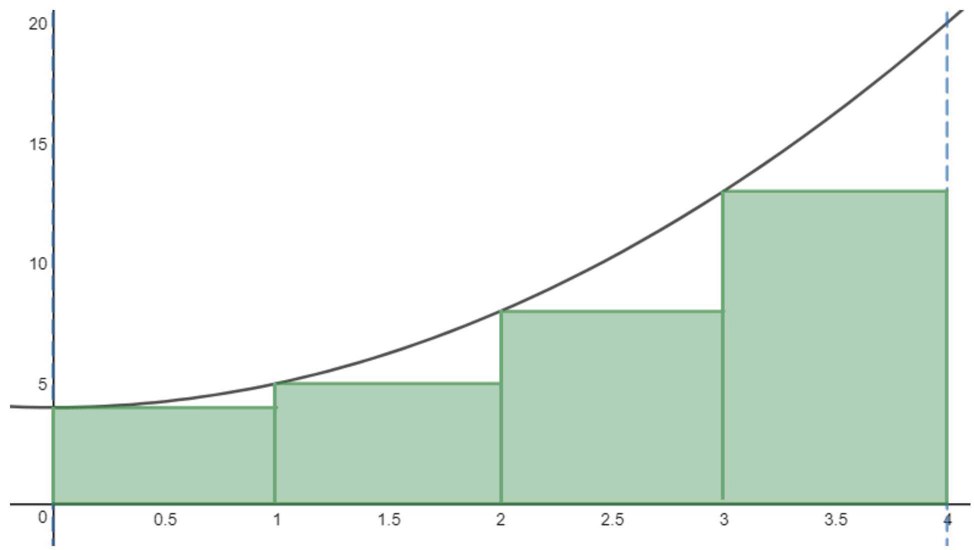
Key Terms

Left hand Sum   Right hand sum

Summary

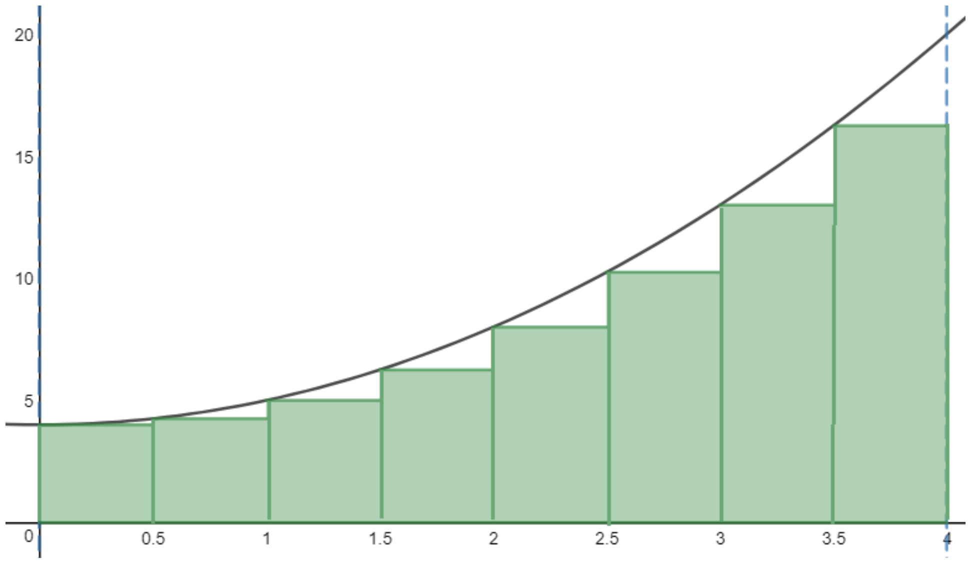
Left and right-hand sums approximate the area under a function's graph and above the x-axis with rectangles. The heights of each rectangle is determined by the function's y value on the left hand side of the rectangle or the right hand side of the rectangle.

Pictured below is a left-hand sum with 4 rectangles from $x = 0$ to $x = 4$.



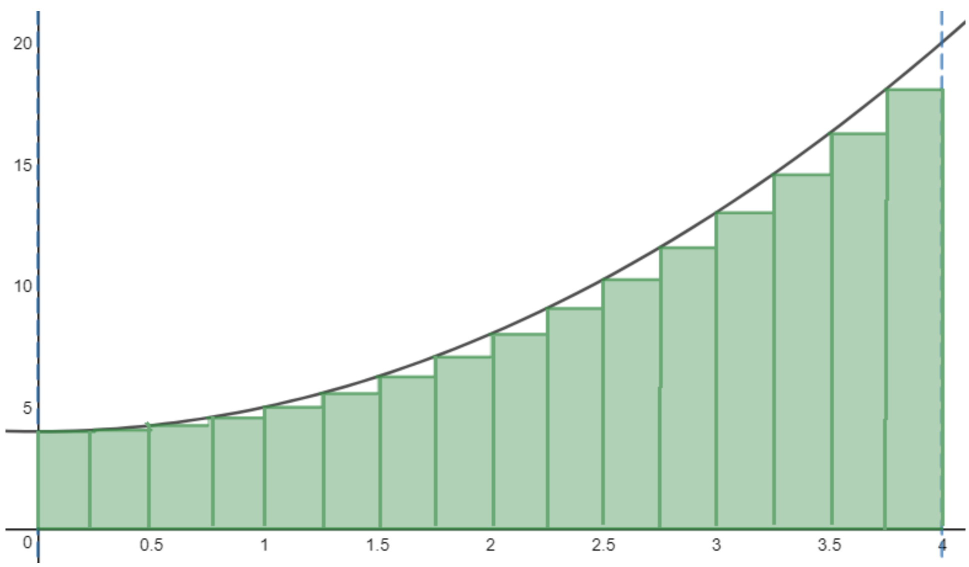
Each rectangle lies below the function's graph, so this estimate would result in an underestimate of the actual area under the function and above the x axis.

Compare this estimate with an estimate obtained from using 8 rectangles.



This estimate also results in an underestimate. But in this case the rectangles "fit" beneath the function resulting in an estimate that is closer to the exact area.

If we increase the number of rectangles to 16, we see that the rectangles fit even better with less of the actual area under the curve missed.



As the number of rectangles increases, the left-hand sum produces an underestimate that gets closer and closer to the exact area under the graph from $x = 0$ to $x = 4$.

A right-hand sum would produce an overestimate that get closer and closer to the exact area as the number of rectangles increases. This means the left and right-hand sums bracket the exact area and get closer and closer to the exact area as the number of rectangles increases.

<u>Notes</u>

Guided Example

Approximate the area under the graph of $f(x)$ and above the x-axis with rectangles from $x = 0$ to $x = 2$, using the following methods with $n = 4$.

$$f(x) = 9 - x^2$$

a. Use left endpoints.

Solution Since we need 4 rectangles from x = 0 to x = 2, each rectangle must be 0.5 wide. Since we are evaluating the function on the left side of the rectangle, the first rectangle is evaluated at $x = 0$. The left sum would be

$$\begin{aligned} \text{Left Sum} &= f(0)\cdot 0.5 + f(0.5)\cdot 0.5 + f(1)\cdot 0.5 + f(1.5)\cdot 0.5 \\ &= 9\cdot 0.5 + 8.75\cdot 0.5 + 8\cdot 0.5 + 6.75\cdot 0.5 \\ &= 16.25 \end{aligned}$$

b. Use right endpoints.

Solution Since we are evaluating the function on the right side of the rectangle, the first rectangle is evaluated at $x = 0.5$. The right sum would be

$$\begin{aligned} \text{Right Sum} &= f(0.5)\cdot 0.5 + f(1)\cdot 0.5 + f(1.5)\cdot 0.5 + f(2)\cdot 0.5 \\ &= 8.75\cdot 0.5 + 8\cdot 0.5 + 6.75\cdot 0.5 + 5\cdot 0.5 \\ &= 14.25 \end{aligned}$$

c. Average the answers in parts a and b.

Solution The average is $\dfrac{16.25 + 14.25}{2} = 15.25$

d. Use midpoints.

Solution Since we are evaluating the function in the middle of the rectangle, the first rectangle is evaluated at $x = 0.25$. The midpoint sum would be

$$\begin{aligned} \text{Midpoint Sum} &= f(0.25)\cdot 0.5 + f(0.75)\cdot 0.5 + f(1.25)\cdot 0.5 + f(1.75)\cdot 0.5 \\ &= 8.9375\cdot 0.5 + 8.4375\cdot 0.5 + 7.4375\cdot 0.5 + 5.9375\cdot 0.5 \\ &= 15.375 \end{aligned}$$

Practice

1. Approximate the area under the graph of $f(x)$ and above the x-axis with rectangles from $x = 0$ to $x = 4$, using the following methods with $n = 4$.

$$f(x) = x^2 + 5$$

a. Use left endpoints.

b. Use right endpoints.

c. Average the answers in parts a and b.

d. Use midpoints.