Section 13.3 The Definite Integral

Question 1 – What is a definite integral?

Question 2 – How is the definite integral related to the approximate area?

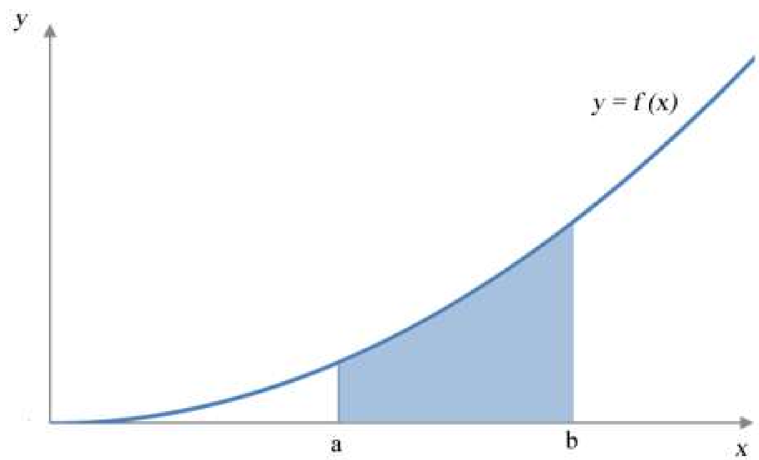
Question 1 – What is a definite integral?

Key Terms

Integral sign Definite integral

Summary

A definite integral is notation for indicating the area between a functions graph and the x axis from one x value to another x value.



The area of the shaded region above would be written as

$$\int_a^b f(x)\,dx$$

The numbers on either end of the integral indicate the x values we want to find the area between. The function between the integral and the dx is the graph we are finding the area under.

If a function's graph is above the x axis, the area between the curve and the x axis is positive. If the function's graph is below the axis, the area between the function and the x axis is negative.

Notes

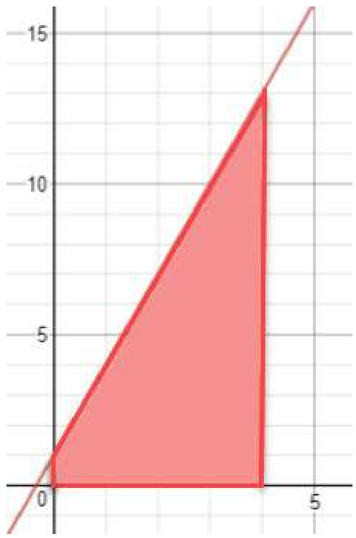
Guided Example

For the definite integral

$$\int_0^4 (3x+1)\,dx$$

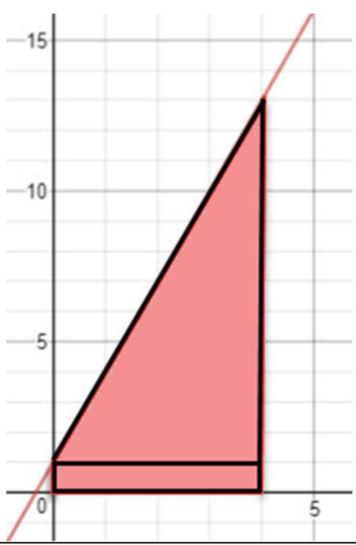
a. Graph the corresponding area.

Solution We are looking for the area from $x = 0$ to $x = 4$ under the curve $y = 3x + 1$.



b. Find the exact value of the area using a geometry formula.

Solution Break the area above into a triangle on top of a rectangle.



Practice

1. For the definite integral

$$\int_0^3 (2x+2)\,dx$$

a. Graph the corresponding area.

b. Find the exact value of the area using a geometry formula.

<table>
<tr>
<td>The rectangle along the bottom has area 4 · 1. The triangle has area $\frac{1}{2}(4)(12)$. The shaded area is the sum of these areas or

$$\int_0^4 (3x+1)\,dx = 4\cdot 1 + \tfrac{1}{2}(4)(12) = 28$$</td>
<td></td>
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<table>
<tr>
<td><u>Guided Example</u></td>
<td><u>Practice</u></td>
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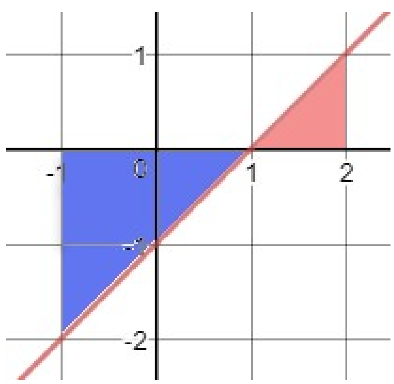
<tr>
<td>For the definite integral
$$\int_{-1}^{2} (x-1)\,dx$$
a. Graph the corresponding area.

Solution Graph y = x - 1 and shade the area between the graph and the x axis from x = - 1 to x = 2.



Since part of this area is below the x axis, it is shaded in a different color than the area above the x axis.

b. Find the exact value of the area using a geometry formula.

Solution The darker area is below the x axis so it must be negative. The area of the darker triangle is $\frac{1}{2}(2)(2)$. The other triangle has area $\frac{1}{2}(1)(1)$. Putting this two together give the value of the definite integral,</td>
<td>2. For the definite integral
$$\int_0^5 (-x+1)\,dx$$
a. Graph the corresponding area.

b. Find the exact value of the area using a geometry formula.</td>
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$$\int_{-1}^{2}(x-1)\,dx=-2+\frac{1}{2}=-1.5$$ The area is negative because more of the shaded area is below the *x* axis than above the *x* axis.	

Question 2 – How is the definite integral related to the approximate area?

Key Terms

Riemann sum Summation

Summary

When we write out the sum of the areas of rectangles, we are computing Riemann sums. The rectangles have heights that are determined by a function. The widths are computed from the starting and ending x values of the region and the number of rectangles, we wish to put in the region.

The area of *n* rectangles under a function $f(x)$ is

$$f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

where Δx is the width of each rectangle. If the first rectangle starts at $x = a$ and ends at $x = b$, the width of each rectangle is

$$\Delta x = \frac{b-a}{n}$$

The values of each x value in the function is determined by whether you are computing the sums using the left side, right side, or midpoint of the rectangle.

This sum can be written more compactly using sigma notation as

$$\sum_{i=1}^{n} f(x_i)\ \Delta x$$

This sum approximates the area of a region under the graph and between two x values. If we use a larger and larger number of rectangles, the approximation becomes better and better. If we were to use an infinite number of rectangles, the area of the rectangles would match the area under the graph exactly. We express this by writing

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\ \Delta x$$

Although we could never put an infinite number of rectangles under a function, we can symbolize it by writing the limit as n approaches infinity.

Notes

Guided Example

Let $f(x) = 2x + 5$ where $x_1 = 1$, $x_2 = 3$, $x_3 = 5$, $x_4 = 7$ and $\Delta x = 2$.

a. Compute $\sum_{i=1}^{4} f(x_i)\Delta x$.

Solution Start by putting in the numbers for the x_i and Δx :

$$\sum_{i=1}^{4} f(x_i)\Delta x = f(1)\cdot 2 + f(3)\cdot 2 + f(5)\cdot 2 + f(7)\cdot 2$$

Now compute the function values and simplify the resulting sum,

$$\sum_{i=1}^{4} f(x_i)\Delta x = 7\cdot 2 + 11\cdot 2 + 15\cdot 2 + 19\cdot 2 = 104$$

b. If the sum on part a corresponds to a left-hand sum, what definite integral is it approximating?

Solution Since the Riemann sum describes a left hand sum, x_1 must be the starting value for the definite integral. The height of the last rectangle comes from $x_4 = 7$. But this is the left side of the rectangle so it must extend to 9. This means the area under $f(x) = 2x + 5$ extends from 1 to 9 and the definite integral is

$$\int_1^9 (2x+5)\,dx$$

Practice

1. Let $f(x) = 5x - 1$ where $x_1 = 3$, $x_2 = 5$, $x_3 = 7$, $x_4 = 9$ and $\Delta x = 2$.

a. Compute $\sum_{i=1}^{4} f(x_i)\Delta x$.

b. If the sum on part a corresponds to a right-hand sum, what definite integral is it approximating?

Notes