Section 14.1 The Substitution Method

Question 1 – How do we find the antiderivative of functions involving compositions?

Question 2 – How is the exact area under a function involving compositions computed?

Question 3 - How do you undo a rate with the Substitution Method?

Question 1 – How do we find the antiderivative of functions involving compositions?

Key Terms

Substitution method

Summary

To understand how the substitution works in computing antiderivatives, we need to first make sure we understand how the chain rule works. Let's start by taking the derivative of

$$y=\left(x^2+2x\right)^7$$

To apply the chain rule to this function, we identify the inside part, g(x), of the right side as

$$g(x)=x^2+2x$$

and the outside part, f(x), as

$$f(x)=x^7$$

This means that the function is being written as a composition in the form $f(g(x))$. The derivatives of these functions are

$$g(x)=x^2+2x \quad \rightarrow \quad g'(x)=2x+2$$

$$f(x)=x^7 \quad \rightarrow \quad f'(x)=7x^6$$

This results in the derivative

$$\frac{dy}{dx}=\underbrace{7\left(x^2+2x\right)^6}_{f'(g(x))}\underbrace{\left(2x+2\right)}_{g'(x)}$$

The corresponding antiderivative would be

$$\int 7\left(x^2+2x\right)^6\left(2x+2\right)dx=\left(x^2+2x\right)^7+C$$

The derivative and antiderivative are opposite processes of each other:

$$\left(x^2+2z\right)^7 \;\underset{\text{u substitution}}{\overset{\text{chain rule}}{\rightleftarrows}}\; 7\left(x^2+2x\right)^6\left(2x+2\right)$$

The opposite process of the chain rule is called u substitution. In this antiderivative technique, the inside function g(x) is called u and is used to simplify the integrand. Let's look at how this is done. We'll find the antiderivative

$$\int 7\left(x^2+2x\right)^6\left(2x+2\right)dx$$

Identify the inside function as $u=x^2+2x$. The derivative of the inside function is $\frac{du}{dx}=2x+2$. Multiply each side by *dx* to give

$$dx\cdot\frac{du}{dx}=\left(2x+2\right)\cdot dx$$

$$du=\left(2x+2\right)dx$$

We can find *u* and *du* in the integrand:

$$\int 7\underbrace{\left(x^2+2x\right)}_{u}{}^6\underbrace{\left(2x+2\right)dx}_{du}=\int 7u^6\,du$$

Notice that this is simply the antiderivative of the outside function, $f'(u)$. We can evaluate this antiderivative with the power rule for antiderivatives,

$$\int 7u^6\,du=u^7+C$$

Since the original variable in the p[problem was x, we need to get back to that variable using $u=x^2+2x$. This means the antiderivative is $\left(x^2+2x\right)^7+C$ or

$$\int 7\left(x^2+2x\right)^6\left(2x+2\right)dx=\left(x^2+2x\right)^7+C$$

Now let's review this all put together:

$$\int 7\underbrace{\left(x^2+2x\right)}_{u}{}^6\underbrace{\left(2x+2\right)dx}_{du} = \int 7u^6\,du$$

$$= 7\int u^6\,du$$

$$= 7\cdot\tfrac{1}{7}u^7 + C$$

$$= u^7 + C$$

$$= \left(x^2+2x\right)^7 + C$$

$$u = x^2+2x \quad \rightarrow \quad dx\cdot\frac{du}{dx} = \left(2x+2\right)dx$$

$$du = \left(2x+2\right)dx$$

<u>Notes</u>

Guided Example

Use substitution to find the indefinite integral:

$$\int 5(5x+1)^4\,dx$$

Solution

$$\int 5(5x+1)^4\,dx = \int \underbrace{(5x+1)}_{u}{}^4\,\underbrace{5\,dx}_{du}$$

$$= \int u^4\,du$$

$$= \tfrac{1}{5}u^5 + C$$

$$= \tfrac{1}{5}(5x+1)^5 + C$$

$u = 5x+1 \quad \rightarrow \quad dx \cdot \dfrac{du}{dx} = 5\,dx$

$du = 5\,dx$

Practice

1. Use substitution to find the indefinite integral:

$$\int 2x\left(x^2+1\right)^3 dx$$

Guided Example

Use substitution to find the antiderivative

$$\int 7\left(x^2+2x\right)^6\left(x+1\right)dx$$

Solution Let $u=x^2+2x$ and $\frac{du}{dx}=2x+2$. If we multiply both sides by ½ on the derivative we get $\frac{1}{2}\frac{du}{dx}=x+1$. This means $\frac{1}{2}du=\left(x+1\right)dx$ and allows us to rewrite the right-hand side of the equation above as

$$\int 7\underbrace{\left(x^2+2x\right)}_{u}^6\underbrace{\left(x+1\right)dx}_{\frac{1}{2}du}=\frac{1}{2}\int 7u^6\,du$$

The antiderivative is found with the power rule as $\frac{1}{2}u^7+C$ so the final solution is

$$\int 7\left(x^2+2x\right)^6\left(x+1\right)dx=\frac{1}{2}\left(x^2+2x\right)^7+C$$

The $\frac{1}{2}$ in the antiderivative balances out the doubling we needed to do to introduce the correct *du*. Now more compactly:

$$\int 7\left(x^2+2x\right)^6\left(x+1\right)=\int 7\underbrace{\left(x^2+2x\right)}_{u}^6\underbrace{\left(x+1\right)dx}_{\frac{1}{2}du}$$

$$=\frac{1}{2}\cdot 7\int u^6du$$

$$=\frac{1}{2}\cdot 7\cdot\frac{1}{7}u^7+C$$

$$=\frac{1}{2}\left(x^2+2x\right)+C$$

$$u=x^2+2x\quad\rightarrow\quad \frac{1}{2}\cdot\frac{du}{dx}=\frac{1}{2}\left(2x+2\right)$$

$$dx\cdot\frac{1}{2}\cdot\frac{du}{dx}=\left(x+1\right)dx$$

$$\frac{1}{2}\cdot du=\left(x+1\right)dx$$

Practice

2. Use substitution to find the antiderivative

$$\int \left(x^3 + x^2\right)^3 \left(6x^2 + 4x\right) dx$$

Guided Example

Use substitution to find the indefinite integral:

$$\int \frac{6x+3}{\left(x^2+x\right)^5}\,dx$$

Solution

$$\int \frac{6x+3}{\left(x^2+x\right)^5}\,dx = \int \frac{1}{\underbrace{\left(x^2+x\right)}_{u}{}^5}\underbrace{(6x+3)\,dx}_{3du}$$

$$u = x^2+x \quad \rightarrow \quad 3\cdot\frac{du}{dx} = 3(2x+1)$$

$$dx\cdot 3\cdot\frac{du}{dx} = (6x+3)\,dx$$

$$3\cdot du = (6x+3)\,dx$$

$$= 3\int \frac{1}{u^5}\,du$$

$$= 3\int u^{-5}\,du$$

$$= 3\frac{u^{-4}}{-4} + C$$

$$= -\frac{3}{4}\frac{1}{\left(x^2+x\right)^4} + C$$

Practice

3. Use substitution to find the indefinite integral:

$$\int \frac{10}{(5x-1)^2}\,dx$$

Guided Example

Evaluate the indefinite integral:

$$\int 4t\, e^{t^2}\, dt$$

Solution

$$\int 4t\, e^{t^2}\, dt = \int e^{\overbrace{t^2}^{u}} \cdot \underbrace{4t\, dt}_{2\,du}$$

$$= 2\int e^{u}\, du$$

$$= 2e^{u} + C$$

$$= 2e^{t^2} + C$$

$$u = t^2 \quad \rightarrow \quad 2 \cdot \frac{du}{dt} = 2 \cdot 2t$$

$$dt \cdot 2 \cdot \frac{du}{dt} = 4t\, dt$$

$$2 \cdot du = 4t\, dt$$

Practice

4. Evaluate the indefinite integral:

$$\int e^{2t+3}\, dt$$

Question 2 – How is the exact area under a function involving compositions computed?

Key Terms

Fundamental Theorem of Calculus

Summary

Area between a function and the x axis is calculated using a definite integral. We can use the Fundamental Theorem of Calculus to find the area,

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is the antiderivative of $f(x)$. If $f(x)$ involves compositions, substitution may be used to find the antiderivative. For instance, in the previous question we used u substitution to find the antiderivative

$$\int 7\left(x^2+2x\right)^6\left(2x+2\right)dx = \left(x^2+2x\right)^7 + C$$

If we want to evaluate the definite integral

$$\int_0^2 7\left(x^2+2x\right)^6\left(2x+2\right)dx$$

Since we know the antiderivative is $F(x) = \left(x^2+2x\right)^7 + C$, the value of the definite integral is

$$\begin{aligned}\int_0^2 7\left(x^2+2x\right)^6\left(2x+2\right)dx &= \left[\left(x^2+2x\right)^7 + C\right]_0^2 \\ &= \left(\left(2^2+2\cdot 2\right)^7 + C\right) - \left(\left(0^2+2\cdot 0\right)^7 + C\right) \\ &= 2097152 + C - C \\ &= 2097152\end{aligned}$$

Notes

Guided Example

Evaluate the definite integral:

$$\int_0^3 (x-1)(x^2-2x)^2\,dx$$

Solution Start by finding the antiderivative by u substitution:

$$\int (x-1)(x^2-2x)^2\,dx = \int \underbrace{(x^2-2x)}_{u}{}^2\,\underbrace{(x-1)\,dx}_{\frac{1}{2}du}$$

$$u = x^2-2x \quad \to \quad \frac{1}{2}\cdot\frac{du}{dx} = \frac{1}{2}\cdot(2x-2)$$

$$dx\cdot\frac{1}{2}\cdot\frac{du}{dx} = (x-1)\,dx$$

$$\frac{1}{2}\cdot du = (x-1)\,dx$$

$$= \frac{1}{2}\int u^2 du$$

$$= \frac{1}{2}\frac{u^3}{3} + C$$

$$= \frac{1}{6}(x^2-2x)^3 + C$$

Now that we know the antiderivative, use the Fundamental Theorem of Calculus to find the definite integral,

$$\int_0^3 (x-1)(x^2-2x)^2\,dx = \left[\frac{1}{6}(x^2-2x)^3 + C\right]_0^3$$

$$= \left(\frac{1}{6}(3^2-2\cdot 3)^3 + C\right) - \left(\frac{1}{6}(0^2-2\cdot 0)^3 + C\right)$$

$$= \frac{27}{6} + C - 0 - C$$

$$= \frac{27}{6}$$

Practice

1. Evaluate the definite integral:

$$\int_0^4 3(6x-1)^3 \, dx$$

Guided Example

Evaluate the definite integral:

$$\int_1^8 \frac{6(\ln x)^2}{x}\,dx$$

Solution Start with u substitution to find the antiderivative:

$$\int \frac{6(\ln x)^2}{x}\,dx = \int \underbrace{(\ln x)}_{u}{}^2 \cdot \underbrace{\frac{6}{x}dx}_{6du}$$

$$= 6\int u^2\,du$$

$$= 6\cdot\frac{u^3}{3} + C$$

$$= 2(\ln x)^3 + C$$

$$u = \ln x \quad \rightarrow \quad 6\cdot\frac{du}{dx} = 6\cdot\frac{1}{x}$$

$$dx\cdot 6\cdot\frac{du}{dx} = \frac{6}{x}dx$$

$$6\cdot du = \frac{6}{x}dx$$

Now use the Fundamental Theorem of Calculus to get the definite integral.

$$\int_e^8 \frac{6(\ln x)^2}{x}\,dx = \Big[2(\ln x)^3 + C\Big]_e^8$$

$$= \left(2(\ln 8)^3 + C\right) - \left(2(\ln e)^3 + C\right)$$

$$= 2(\ln 8)^3 + C - 2 - C$$

$$= 2(\ln 8)^3 - 2$$

Practice

2. Evaluate the definite integral:

$$\int_{2}^{4} \frac{2}{(x-1)^2}\,dx$$

Question 3 – How do you undo a rate with the Substitution Method?

Key Terms

Substitution method Fundamental Theorem of Calculus

Summary

If the integrand is given as a composition, we may need to use the Substitution Method to find the antiderivative of the rate. Undoing the rate will reveal the quantity associated with the rate. For instance, undoing the rate at which revenue changes (the marginal revenue) will lead to the revenue function. Applying the Fundamental Theorem of Calculus in this context allows us to compute the total change in revenue,

$$\int_a^b R'(x)\,dx = R(b) - R(a)$$

where $x = a$ and $x = b$ are two different production levels.

Notes

Guided Example

A company has found that the rate at which profit is changing (in millions of dollar per year) is given by the function

$$P'(t) = (8t+8)(t^2+2t+1)^{1/3}$$

a. Find the total profit over the first three years.

Solution To find the total profit over the first three years, we need to compute $P(3) - P(0)$. We can find $P(t)$ from the antiderivative of $P'(t)$. In fact, applying the Fundamental Theorem of Calculus, the total change in profit is computed from the definite integral,

$$\int_0^3 P'(t)\,dt = P(3) - P(0)$$

This is equivalent to

$$\int_0^3 (8t+8)(t^2+2t+1)^{1/3}\,dt = P(3) - P(0)$$

We will accomplish this by applying u substitution to the definite integral:

$$\begin{aligned}
\int (8t+8)(t^2+2t+1)^{1/3}\,dt &= \int \underbrace{(t^2+2t+1)}_{u}{}^{1/3}\,\underbrace{(8t+8)\,dt}_{4du} \\
&= 4\int u^{1/3}\,du \\
&= 4\cdot\frac{u^{4/3}}{4/3} + C \\
&= 3(t^2+2t+1)^{4/3} + C
\end{aligned}$$

Now apply the Fundamental Theorem of Calculus to give

$$\begin{aligned}
\int_0^3 (8t+8)(t^2+2t+1)^{1/3}\,dt &= \left[3(t^2+2t+1)^{4/3} + C\right]_0^3 \\
&= \left(3(3^2+2\cdot3+1)^{4/3} + C\right) - \left(3(0^2+2\cdot0+1)^{4/3} + C\right) \\
&= 3(16)^{4/3} + C - 3(1)^{4/3} - C \\
&\approx 117.95
\end{aligned}$$

in millions of dollars.

b. Find the profit in the fourth year of operation.

Solution

$$\int_3^4 (8t+8)\left(t^2+2t+1\right)^{1/3}\,dt = \left[3\left(t^2+2t+1\right)^{4/3}+C\right]_3^4$$

$$= \left(3\left(4^2+2\cdot 4+1\right)^{4/3}+C\right)-\left(3\left(3^2+2\cdot 0+1\right)^{4/3}+C\right)$$

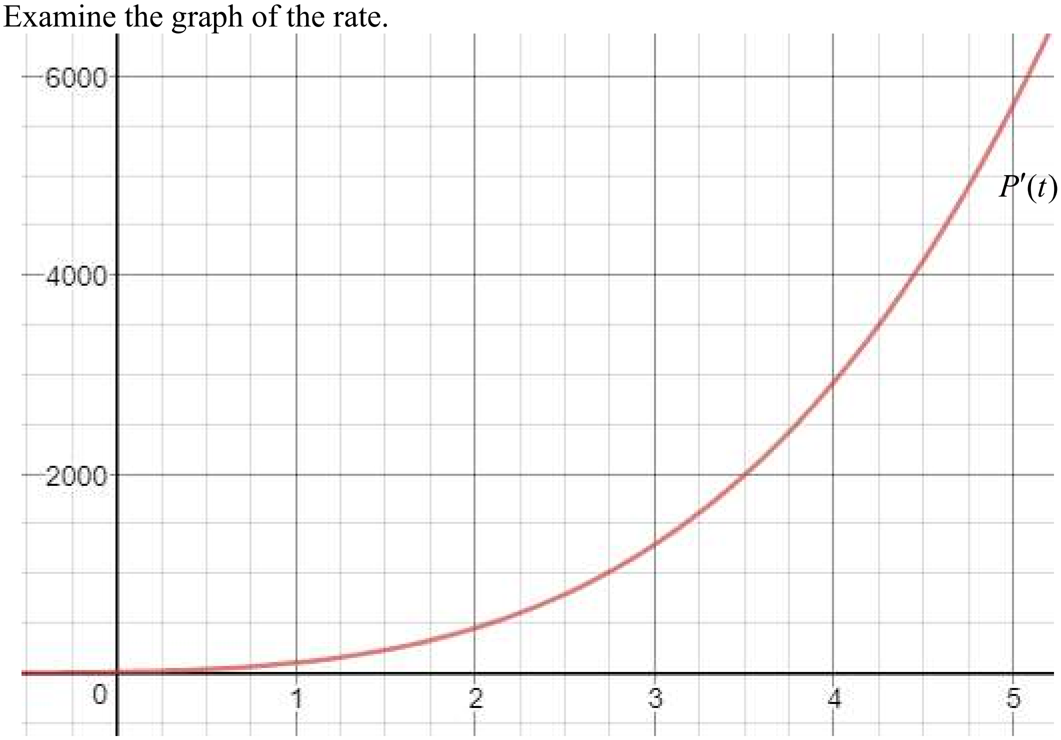
$$= 3(25)^{4/3}+C-3(16)^{4/3}-C$$

$$\approx 98.35$$

in millions of dollars.

c. What is happening to the annual profit over the long run?

Solution Examine the graph of the rate.



The annual areas from 1 to 2, 2 to 3, 3 to 4, … are getting larger and larger so the annual profit is getting larger and larger.

Practice

1. Suppose a company spends R million dollars on research annually. The rate $S'(R)$ at which sale are increasing at a company is given by

   $$S'(R) = 50e^{0.05R} \text{ dollars of sales per dollars of research}$$

   If the annual research budget is increased from 3 million dollars to 4 million dollars, how much will sales increase?