Section 14.2 Integration by Parts

Question 1 – How do we find the antiderivative of functions involving products?

Question 2 – How is the exact area under a function involving products computed?

Question 1 – How do we find the antiderivative of functions involving products?

Key Terms

Product

Summary

The Integration by Parts Rule allows us to take the antiderivative of functions that can be written as products.

Integration by Parts Rule

If u and v are differentiable functions,

$$\int uv'\,dx = uv - \int vu'\,dx$$

This rule requires us to match the integrand with the two pieces of the product, u and v'. Applying the rule results in a new antiderivative which should be easier to evaluate than the original antiderivative.

For example, consider the antiderivative $\int xe^x dx$ .To apply the Integration by Parts Rule, match u with x and v' with e^x. To find u', we need to take a derivative. To find v, we need to take an antiderivative.

$$u = x \xrightarrow{\text{Derivative}} u' = 1$$
$$v' = e^x \xrightarrow{\text{Antiderivative}} v = e^x$$

Put these pieces into the rule to give

$$\int \underbrace{x}_{u}\underbrace{e^x}_{v'}\,dx = \underbrace{x}_{u}\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\cdot\underbrace{1}_{u'}\,dx$$

The new integral is simply the antiderivative of e^x which is $e^x + C$. The original antiderivative may be evaluated as

$$\int xe^x\,dx = xe^x - e^x + C$$

We can abbreviate these steps as shown below:

$$\int \underbrace{x}_{u}\underbrace{e^x}_{v'}\,dx = \underbrace{x}_{u}\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\cdot\underbrace{1}_{u'}\,dx$$
$$= xe^x - \quad e^x + C$$

$$u = x \xrightarrow{\text{Derivative}} u' = 1$$
$$v' = e^x \xrightarrow{\text{Antiderivative}} v = e^x$$

<u>Notes</u>

Guided Example

Use integration by parts to evaluate the integral

$$\int 8x\,e^{2x}dx$$

Solution

$$\int \underbrace{8x}_{u}\underbrace{e^{2x}}_{v'}\,dx = \underbrace{8x}_{u}\cdot\underbrace{\tfrac{1}{2}e^{2x}}_{v} - \int \underbrace{\tfrac{1}{2}e^{2x}}_{v}\cdot\underbrace{8}_{u'}\,dx$$
$$= 4xe^{2x} - 2e^{2x} + C$$

$$u = 8x \xrightarrow{\text{Derivative}} u' = 8$$
$$v' = e^{2x} \xrightarrow{\text{Antiderivative}} v = \tfrac{1}{2}e^{2x}$$

The new integral, $\int \frac{1}{2}e^{2x}\cdot 8\,dx$, is evaluated using the Substitution Rule:

$$\int \tfrac{1}{2}e^{2x}\cdot 8\,dx = 4\int e^{2x}\,dx$$
$$= 4\cdot\tfrac{1}{2}\int e^{u}\,du$$
$$= 2e^{u} + C$$
$$= 2e^{x} + C$$

$$u = 2x \quad\rightarrow\quad \frac{1}{2}\cdot\frac{du}{dx} = \frac{1}{2}\cdot 2$$
$$dx\cdot\frac{1}{2}\cdot\frac{du}{dx} = dx$$
$$\frac{1}{2}\cdot du = dx$$

Practice

1. Use integration by parts to evaluate the integral

$$\int 10x\,e^{4x}dx$$

Guided Example

Use integration by parts to evaluate the integral

$$\int 2x\ln(4x)\,dx$$

Solution

$$\int \underbrace{2x}_{v'}\underbrace{\ln(4x)}_{u}\,dx = \underbrace{\ln(4x)}_{u}\cdot\underbrace{x^2}_{v} - \int \underbrace{x^2}_{v}\cdot\underbrace{\tfrac{1}{x}}_{u'}\,dx$$

$$= \ln(4x)\cdot x^2 - \int x\,dx$$

$$= \ln(4x)\cdot x^2 - \frac{x^2}{2} + C$$

$$u = \ln(4x) \xrightarrow{\text{Derivative}} u' = \frac{1}{4x}\cdot 4 = \frac{1}{x}$$

$$v' = 2x \xrightarrow{\text{Antiderivative}} v = x^2$$

Practice

2. Use integration by parts to evaluate the integral

$$\int 3x^2\ln(10x)\,dx$$

Guided Example

Evaluate the integral

$$\int x \ln(\sqrt{x})\, dx$$

Solution The key to this problem is to rewrite the square root as a power and then to utilize the logarithm rule that allows powers to be brought out as a multiplier:

$$\int x \ln(\sqrt{x})\, dx = \int x \ln(x^{1/2})\, dx$$

$$= \int \tfrac{1}{2} x \ln(x)\, dx$$

Now we can apply integration by parts:

$$\int \underbrace{\ln(x)}_{u} \underbrace{\tfrac{1}{2}x}_{v'}\, dx = \underbrace{\ln(x)}_{u} \underbrace{\tfrac{x^2}{4}}_{v} - \int \underbrace{\tfrac{x^2}{4}}_{v} \cdot \underbrace{\tfrac{1}{x}}_{u'}\, dx$$

$$= \ln(x) \cdot \frac{x^2}{4} - \int \frac{x}{4}\, dx$$

$$= \ln(x) \cdot \frac{x^2}{4} - \frac{x^2}{8} + C$$

$$u = \ln(x) \xrightarrow{\text{Derivative}} u' = \frac{1}{x}$$

$$v' = \frac{1}{2}x \xrightarrow{\text{Antiderivative}} v = \frac{x^2}{4}$$

Practice

3. Evaluate the integral

$$\int 2x \ln(x^5)\, dx$$

Question 2 – How is the exact area under a function involving products computed?

Key Terms

Product — Integration by Parts — Fundamental Theorem of Calculus

Summary

When we find the exact area under a function by evaluating the definite integral, we utilize the Fundamental Theorem of Calculus. The Fundamental Theorem of Calculus helps us to evaluate the definite integral,

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is the antiderivative of $f(x)$. If $f(x)$ involves products, integration by parts may be used to find the antiderivative.

For instance, to find the area between $y = xe^x$ and the x axis from $x = 0$ to $x = 3$ we must evaluate the definite integral

$$\int_0^3 xe^x\, dx$$

In the previous question, we used integration by parts to find the antiderivative:

$$\int xe^x\, dx = xe^x - e^x + C$$

To evaluate the definite integral, we simply substitute x = 3 and x = 0 into the antiderivative and subtract,

$$\begin{aligned}\int_0^3 xe^x\, dx &= \left[xe^x - e^x + C\right]_0^3 \\ &= \left(3e^3 - e^3 + C\right) - \left(0e^0 - e^0 + C\right) \\ &= 2e^3 + 1 \\ &\approx 41.17\end{aligned}$$

Notes

Guided Example

Evaluate the definite integral

$$\int_1^{10} \ln(2x)\,dx$$

Solution To find the antiderivative of ln(2x), think of the integrand as the product of 1 and ln(2x). Then apply integration by parts to this product:

$$\int \underbrace{\ln(2x)}_{u}\cdot\underbrace{1}_{v'}\,dx = \underbrace{\ln(2x)}_{u}\,\underbrace{x}_{v} - \int \underbrace{x}_{v}\cdot\underbrace{\tfrac{1}{x}}_{u'}\,dx$$

$$= \ln(2x)\cdot x - \int 1\,dx$$

$$= \ln(2x)\cdot x - x + C$$

$$u = \ln(2x) \xrightarrow{\text{Derivative}} u' = \frac{1}{2x}\cdot 2 = \frac{1}{x}$$

$$v' = 1 \xrightarrow{\text{Antiderivative}} v = x$$

Note that the derivative of u is completed with the Chain Rule. Now that we have the antiderivative, we can apply the Fundamental Theorem of Calculus to evaluate the definite integral.

$$\int_1^{10} \ln(2x)\,dx = \left[\ln(2x)\cdot x - x + C\right]_1^{10}$$

$$= \left(\ln(20)\cdot 10 - 10 + C\right) - \left(\ln(2)\cdot 1 - 1 + C\right)$$

$$= \ln(20)\cdot 10 - 9 - \ln(2)$$

$$\approx 20.26$$

Practice

1. Evaluate the definite integral

$$\int_1^{3} \ln(5x)\,dx$$