Section 12.1 Extrema of a Function

Question 1 – What is the difference between a relative extremum and an absolute extremum?

Question 2 – What is a critical point of a function?

Question 3 - How do you find the relative extrema of a function?

Question 4 - How do you find the absolute extrema of a function?

Question 1 – What is the difference between a relative extremum and an absolute extremum?

Key Terms

Relative maximum	Relative minimum
Local maximum	Local minimum
Extrema	Extreme Value Theorem

Summary

Extrema are the high and low points of a function. The high and low points may be relative maximum and relative minimum (also called local maximum and local minimum). These points are the bumps and dips on a function that are higher or lower than the points nearby.

Absolute maximum and absolute minimums are the very highest and lowest points on a graph. They may occur at a relative extremum or at the endpoints of an interval. The Extreme Value Theorem give the conditions under which absolute extrema are guaranteed.

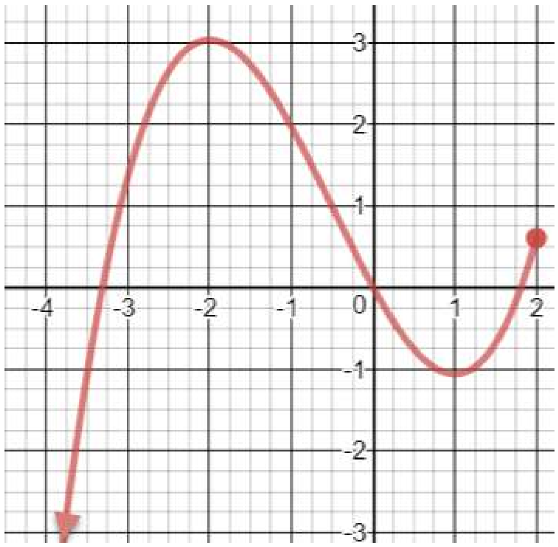
> **Extreme Value Theorem**
>
> A function f that is continuous on a closed interval is guaranteed to have both an absolute maximum and an absolute minimum.

The key ingredient to this theorem is that the function must be continuous on a closed interval (include the endpoints). Only in this situation are we guaranteed absolute extrema. Without these conditions, there is no guarantee. An absolute extremum may or may not occur when the function is discontinuous or on an open interval (not including one of both endpoints).

Notes

Guided Example

For the function graphed below:



a. Locate the coordinates of all relative extrema.

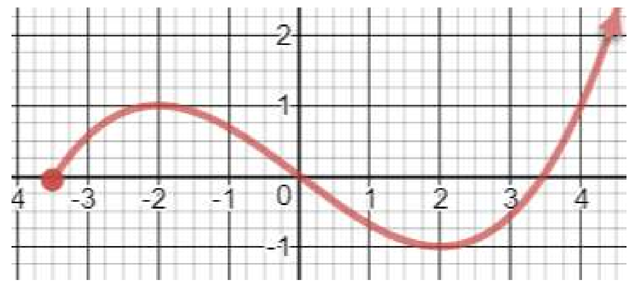
Solution Relative extrema are the locations of bumps and dips in the function. These bumps and dips are higher or lower than points nearby. On this function, there is a relative maximum at (-2, 3) and a relative minimum at (1, -1).

b. Locate the coordinates of all absolute extrema.

Solution Absolute extrema are the very highest and lowest points on the graph. They may occur at a relative extremum or at the endpoints of a function. Since the function is not defined on a closed interval, absolute extrema are not guaranteed. For this function, the absolute maximum is at (-2, 3). There is no absolute minimum since the function continues downward on the left side of the function.

Practice

1. For the function graphed below:

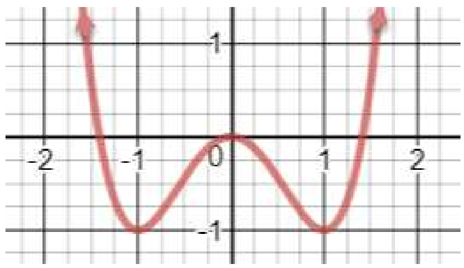


a. Locate the coordinates of all relative extrema.

b. Locate the coordinates of all absolute extrema.

Guided Example

For the function graphed below:



a. Locate the coordinates of all relative extrema.

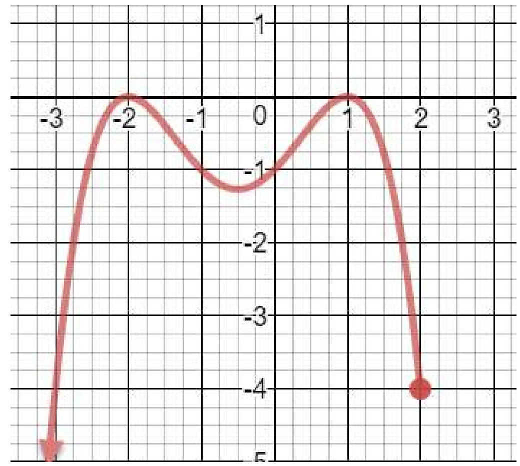
Solution There is a relative maximum at (0, 0) and relative minimums at (-1, -1) and (1, -1).

b. Locate the coordinates of all absolute extrema.

Solution The function is defined on an open interval so no absolute extrema are guaranteed. However, the very lowest points on the graph are at (-1, -1) and (1, -1) so these are absolute minimum. This function has no absolute maximum.

Practice

For the function graphed below:



a. Locate the coordinates of all relative extrema.

b. Locate the coordinates of all absolute extrema.

Question 2 – What is a critical point of a function?

Key Terms

Critical point	Critical number
Increasing function	Decreasing function

Summary

Critical points on a function are where the function's derivative is equal to zero or undefined. On the graph, this typically corresponds to places where the function's tangent line has a slope of zero or the slope of the tangent line is undefined.

Critical points are helpful because they break up where a function in increasing or decreasing. A function that is increasing rises as we move from left to right on the graph. A function that is decreasing drops as we move from left to right on the graph. To help us determine where a function is increasing and decreasing, we use the sign of the derivative.

Test for Determining the Intervals Where a Function is Increasing or Decreasing

Suppose a function f is defined on an open interval and that we are able to compute the derivative of the function at each point in the open interval.

- If $f'(x) > 0$ at every x value in the interval, then f is increasing on the interval.
- If $f'(x) < 0$ at every x value in the interval, then f is decreasing on the interval.
- If $f'(x) = 0$ at every x value in the interval, then f is constant on the interval.

We can apply this test by following the strategy below.

Strategy for Tracking the Sign of the Derivative

1. Take the derivative of the function and use it to find all of the critical values. Below a number line, label these values. Above the number line, write = 0 to indicate critical values where the derivative is zero or write "und" to indicate critical values where the derivative is undefined. The open intervals between these values are where we will determine the sign of the derivative.
2. Pick a value in each of the open intervals between the critical values. Substitute these values in $f'(x)$ to determine whether the derivative is positive or negative at these values. Label the number line + or – to indicate whether the derivative is positive or negative.
3. The function is increasing over the intervals where $f'(x) > 0$ and decreasing over the intervals where $f'(x) < 0$.

Notes

Guided Example

For the function,

$$f(x) = 2x^3 - 9x^2 + 7$$

Locate the intervals where the function is increasing and decreasing.

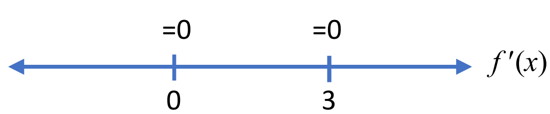
Solution Start by finding the critical points on the function. The derivative of the function is

$$f'(x) = 6x^2 - 18x$$

This function is always defined, so the only place the critical points are is where $f'(x) = 0$. Set the derivative equal to zero and solve for x:

$$6x^2 - 18x = 0$$
$$6x(x-3) = 0$$
$$6x = 0 \quad x - 3 = 0$$
$$x = 0 \quad x = 3$$

Label these critical points on a number line.



Now let's test between the critical points at x = -1, 1, and 5. It is easiest to use the derivative in factored form to do this,

$$f'(x) = 6x\left(x-3\right)$$

For instance, if we put x = -1 into the derivative, the first factor is negative and the second factor is negative,

$$f'(-1) = (-)(-) = +$$

So, the product is positive. Label the number line with a plus to the left of x = 0 along with the

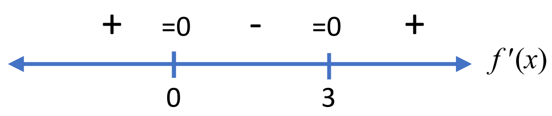
Practice

1. For the function,

$$f(x) = 3x^4 + 8x^3 - 12$$

Locate the intervals where the function is increasing and decreasing.

signs of the derivative in the other section. This will give us the following number line.



This tells us the function is increasing on the intervals (-∞, 0) and (3, ∞) and decreasing on the interval (0, 3).

Guided Example

For the function,

$$f(x) = \frac{x+5}{x+2}$$

Locate the intervals where the function is increasing and decreasing.

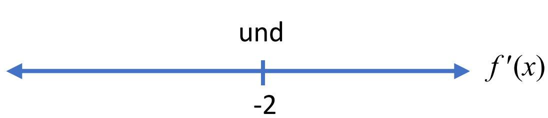
Solution Start by finding the critical points on the function. The derivative of the function is

$$f'(x) = \frac{(x+2)(1)-(x+5)(1)}{(x+2)^2}$$

$$= \frac{-3}{(x+2)^2}$$

This function is undefined at x = -2. Since the numerator is never equal to zero, there are numbers where the derivative is zero.

Label the critical point on a number line.



Now let's test between the critical points at x = -3 and 0.

Practice

2. For the function,

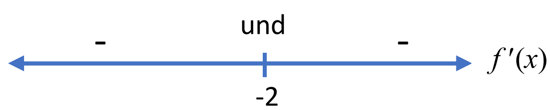
$$f(x) = \frac{x-5}{x-1}$$

Locate the intervals where the function is increasing and decreasing.

If we put x = -3 into the derivative, the numerator is negative and the denomonator is positive,

$$f'(-3) = \frac{(-)}{(+)} = -$$

So, the quotient is negative. Label the number line with a minus to the left of x = -3 along with the signs of the derivative in the other section. This will give us the following number line.



This tells us the function is decreasing on the intervals (-∞, -2) and (-2, ∞). It is not decreasing everywhere since the original function is not defined at x = -2.

Question 3 – How do you find the relative extrema of a function?

Key Terms

First Derivative Test

Summary

We can find the location of relative extrema using the First Derivative Test. This test examines how the derivative changes sign on either side of a critical number to deduce whether it is a relative maximum, relative minimum, or neither.

First Derivative Test

Let f be a non-constant function that is defined at a critical value $x = c$.

- If f' changes from positive to negative at $x = c$, then a relative maximum occurs at the critical point .
- If f' changes from negative to positive at $x = c$, then a relative minimum occurs at the critical point .
- If f' does not change sign at $x = c$, then there is no relative extrema at the corresponding critical point.

The easiest way to apply this test is to examine a derivative number line like the ones we used in the last question.

Notes

Guided Example

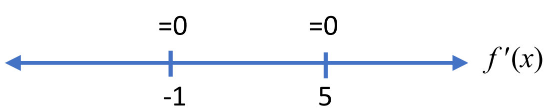
Find the x values at which the function below has relative extrema. Find the values of the relative extrema.

$$h(x) = x^3 - 6x^2 - 15x + 1$$

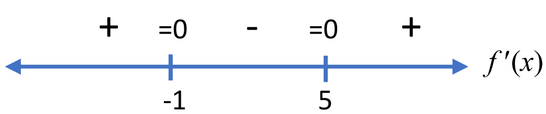
Solution To find the critical numbers, find the derivative,

$$\begin{aligned} h'(x) &= 3x^2 - 12x - 15 \\ &= 3\left(x^2 - 4x - 5\right) \\ &= 3\left(x - 5\right)\left(x + 1\right) \end{aligned}$$

The derivative was factored to make it easier to find the critical numbers and to make it easier to determine the sign of the derivative. The derivative is equal to zero at $x = 5$ and -1. Label these numbers on the derivative number line.



Now test the derivative at some numbers on either side of the derivative like $x = -2$, 0, and 6.



Examine how the derivative changes on either side of the critical numbers. At $x = -1$, the derivative changes from positive to negative so this critical number corresponds to a relative maximum. At $x = 5$, the derivative changes from negative to positive so this critical number corresponds to a relative minimum.

To compute the corresponding maximum and minimum values, put the critical point into the original function:

$$h(-1) = \left(-1\right)^3 - 6\left(-1\right)^2 - 15\left(-1\right) + 1 = 9$$

$$h(5) = \left(5\right)^3 - 6\left(5\right)^2 - 15\left(5\right) + 1 = -99$$

Practice

1. Find the x values at which the function below has relative extrema. Find the values of the relative extrema.

$$h(x) = 4x^3 - 3x^2 - 18x + 10$$

Guided Example

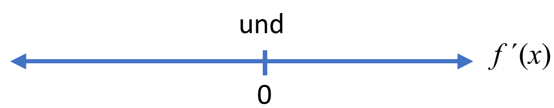
Find the x values at which the function below has relative extrema. Find the values of the relative extrema.

$$h(x) = \frac{2x^2 - 128}{x}$$

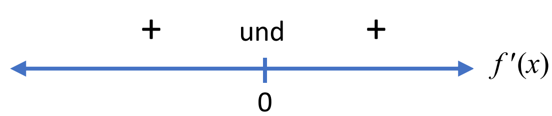
Solution To find the critical numbers, find the derivative with the quotient rule,

$$h'(x) = \frac{x(4x) - (2x^2 - 128)}{x^2}$$
$$= \frac{2x^2 + 128}{x^2}$$

The critical numbers are where the derivative is equal to zero (numerator equal to zero) or where the derivative is undefined (denominator equal to zero). For this function, there are no numbers on the number line where the numerator is equal to zero. However, the denominator is equal to zero at $x = 0$.



Now test the derivative at some numbers on either side of the derivative like $x = -1$ and 1.



Notice how the derivative does not change on either side of the critical numbers. Even if it did it would still not be a relative extremum since the original function is not defined at $x = 0$.

Practice

2. Find the x values at which the function below has relative extrema. Find the values of the relative extrema.

$$h(x) = \frac{x^2}{x - 1}$$

Question 4 – How do you find the absolute extrema of a function?

Key Terms

Critical number

Absolute maximum

Absolute minimum

Summary

The absolute extrema of a function is the very highest and lowest points on a function. The Extreme Value Theorem tells us that we are guaranteed an absolute maximum and absolute minimum if the function is continuous over a closed interval. In this case, the absolute extrema will occur at the critical points or endpoints of the interval. To distinguish which points are the absolute extrema, substitute the points into the original function and determine which point is highest (the absolute maximum) and which point is lowest (the absolute minimum).

Notes

Guided Example

Find the absolute maximum and absolute minimum values of

$$y = x^3 - 3x^2 + 7$$

over the interval [-1, 4] and the *x* values at which they occur.

Solution The absolute extrema will occur at the critical numbers or the endpoints of the interval [-1, 4]. Let's start by computing the derivative so that we can find the critical numbers.

$$\frac{dy}{dx} = 3x^2 - 6x$$
$$= 3x(x-2)$$

Setting each of these factors equal to zero give critical numbers at $x = 0$ and 2.

The absolute extrema will occur at these numbers or $x = -1$ or 4. To decide which is highest and lowest on the function, construct a table with these values and the corresponding y values.

x	*y*
-1	3
0	7
2	3
4	23

There are two absolute minimum at $x = -1$ and 2 with minimum values of 3. The absolute maximum is at $x = 4$ with maximum value of 23.

Practice

1. Find the absolute maximum and absolute minimum values of

$$y = x^4 - 8x^3 + 10x^2 + 22$$

over the interval [0, 10] and the *x* values at which they occur.

Section 12.2 Higher Derivatives

Question 1 – How is the second derivative calculated?

Question 2 – How are other higher derivatives calculated?

Question 3 - What does the second derivative of a function tell you about a function?

Question 4 - What is the point of diminishing returns?

Question 1 – How is the second derivative calculated?

Key Terms

Second derivative

Summary

The second derivative is computed by taking another derivative of the first derivative. This may be represented as $f''(x)$ where the two primes indicate the second derivative. The second derivative may also be written as $\frac{d^2y}{dx^2}$.

Notes

Guided Example

Practice

Guided Example	Practice
Find the second derivative of $$y = 3x^4 - 2x^3 + 5x^2 - 7x + 4$$ **Solution** Start by taking the derivative of each term: $$\frac{dy}{dx} = 12x^3 - 6x^2 + 10x - 7$$ To find the second derivative, take the derivative of the first derivative, $$\frac{d^2y}{dx^2} = 36x^2 - 12x + 10$$	1. Find the second derivative of $$y = -6x^5 - x^2 + 7x + 10$$

Guided Example

Practice

Guided Example	Practice
Find the second derivative of $f(x) = \dfrac{x-1}{x+2}$. **Solution** Start by finding the first derivative. Since this function is a quotient, we'll need to apply the quotient rule with $u = x-1$ and $v = x+2$. Taking the derivatives of these pieces yields $$\begin{matrix} u = x-1 \\ v = x+2 \end{matrix} \quad \rightarrow \quad \begin{matrix} u' = 1 \\ v' = 1 \end{matrix}$$ Put these pieces into the quotient rule give us $$f'(x) = \frac{(x+2)\cdot 1 - (x-1)\cdot 1}{(x+2)^2}$$ Now simplify the first derivative. Before we take another derivative, we should simplify the first derivative as much as possible. This will make the second derivative a lot easier. In this case, let's remove the parentheses and distribute the subtraction to give $$\frac{(x+2)\cdot 1 - (x-1)\cdot 1}{(x+2)^2} = \frac{3}{(x+2)^2}$$	2. Find the second derivative of $f(x) = \dfrac{x+1}{x-3}$.

To make the next derivative easier to do rewrite the first derivative. We could use the quotient rule again, but it is easier to rewrite the first derivative as

$$f'(x) = 3\left(x+2\right)^{-2}$$

We'll need the chain rule to take the second derivative with

$$\begin{array}{c} \text{inside} = x+2 \\ \text{outside} = 3x^{-2} \end{array} \rightarrow \begin{array}{c} \text{inside}' = 1 \\ \text{outside}' = -6x^{-3} \end{array}$$

Applying the chain rule yields

$$f''(x) = -6\left(x+2\right)^{-3} \cdot 1$$

We can rewrite this as

$$f''(x) = \frac{-6}{\left(x+2\right)^{3}}$$

Question 2 – How are other higher derivatives calculated?

Key Terms

Third derivative Fourth derivative

Summary

If we continue to take the derivative of the second derivative, we will get the third, fourth, or fifth derivative. In fact, you can continue the process of taking the derivative to get even higher derivatives. As we get the higher derivatives, we denote them depending on how the function was given.

Original	First Derivative	Second Derivative	Third derivative	Fourth Derivative	n^{th} Derivative
$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$	$f^{(n)}(x)$
y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$	$\frac{d^4y}{dx^4}$	$\frac{d^ny}{dx^5}$

You may also see a slightly different notation that indicates that you should take the second derivative. In this notation, the symbol $\frac{d^2}{dx^2}$ means take the second derivative of whatever follows in brackets. So we might write $\frac{d^2}{dx^2}\left[x^3+x\right]$ when we want to take the second derivative of x^3+x .

Notes

Guided Example	Practice
Find the third and fourth derivatives of $$y = 3x^4 - 2x^3 + 5x^2 - 7x + 4$$ **Solution** In a previous example we calculated the second derivative to be $$\frac{d^2y}{dx^2} = 36x^2 - 12x + 10$$ We can compute the third derivative by taking another derivative $$\frac{d^3y}{dx^3} = 72x - 12$$ And another derivative to get the fourth derivative, $$\frac{d^4y}{dx^4} = 72$$	1. Find the third and fourth derivatives of $$y = -6x^5 - x^2 + 7x + 10$$

Guided Example	Practice
Find the third and fourth derivatives of $$f(x) = \frac{x-1}{x+2}$$ **Solution** In a previous example, we found the second derivative to be $$f''(x) = -6(x+2)^{-3}$$ To find the third derivative, apply the chain rule again to get $$f^{(3)}(x) = 18(x+2)^{-4}$$ And another time to get $$f^{(4)}(x) = -72(x+2)^{-5}$$	2. Find the third and fourth derivatives of $$f(x) = \frac{x+1}{x-3}$$

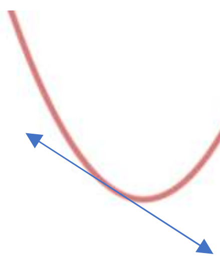
Question 3 – What does the second derivative of a function tell you about a function?

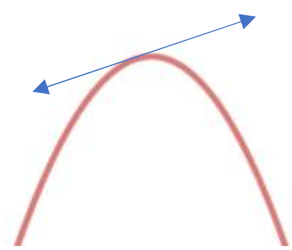
Key Terms

Concavity | Concave up
Concave down | Point of inflection

Summary

The second derivative of a function is related to how the graph bends. A graph that is concave up lies above its tangent lines. A graph that is concave down lies below its tangent lines.





Concave Up Concave down

Concavity and the Second Derivative

- When the graph of a function $f(x)$ is concave up, the second derivative $f''(x)$ is positive.
- When the graph of a function $f(x)$ is concave down, the second derivative $f''(x)$ is negative.

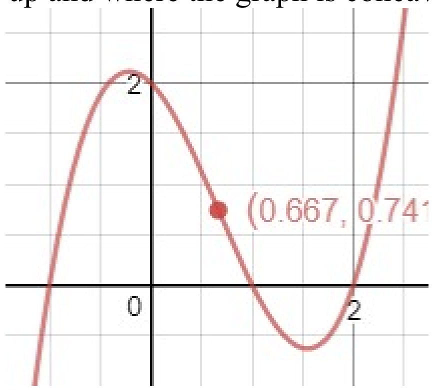
We can create a number line for the second derivative to track where the graph is concave up or concave down. By examining the number line, we can determine where the graph changes from concave up to concave down or concave down to concave up. Points where the concavity changes are called points of inflection.

The second derivative is also useful for deciding whether critical points correspond to relative maxima or relative minima. If the sign of the second derivative is positive at a critical point, then the graph is concave up at that point. This means the critical point is a relative minimum. If the sign of the second derivative is negative at a critical point, then the graph is concave down at that point. This means the critical point is a relative maximum.

Notes

Guided Example

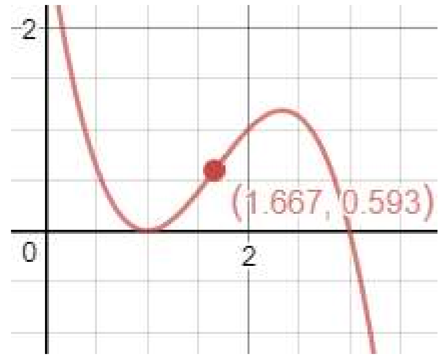
Find the open intervals where the graph is concave up and where the graph is concave down.



Solution The graph is concave down when it lies below the tangent lines from (-∞, 0.667). The graph is concave up when the graph lies above the tangent lines from (0.667, ∞).

Practice

1. Find the open intervals where the graph is concave up and where the graph is concave down.



Guided Example

Use a second derivative number line to determine the open intervals where

$$f(x) = x^3 - 6x^2 + x + 9$$

is concave up and concave down.

Solution To construct a second derivative number line, we need the second derivative,

$$f'(x) = 3x^2 - 12x + 1$$

$$f''(x) = 6x - 12$$

Set the second derivative equal to zero and solve for x:

$$6x - 12 = 0$$

$$6x = 12$$

$$x = 2$$

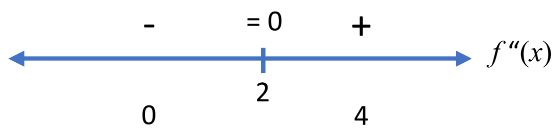
Label this number on the number line. Now we need to test the second derivative on either side of $x = 2$.

Practice

2. Use the second derivative number line to determine the open intervals where

$$f(x) = 2x^3 + 6x^2 + 9x + 7$$

is concave up and concave down.



Since the second derivative is negative on the left side of x = 2, the graph is concave down in the open interval (-∞, 2). The graph is concave up in the open interval (2, ∞) since the second derivative is positive there.

Guided Example

Use a second derivative to test whether each critical point of

$$f(x) = x^3 - 6x^2 + 9$$

is a relative maximum or relative minimum.

Solution To find the critical numbers, find the first derivative,

$$f'(x) = 3x^2 - 12x$$

Set the derivative equal to zero and factor:

$$3x^2 - 12x = 0$$

$$3x(x-4) = 0$$

$$x = 0 \quad x = 4$$

To apply the second derivative test, compute the second derivative,

$$f''(x) = 6x - 12$$

Put each critical point into the second derivative:

$$f''(0) = 6(0) - 12 = -12$$

$$f''(4) = 6(4) - 12 = 12$$

Since the second derivative at $x = 0$ is negative, the graph is concave down there, so it is a relative maximum. At $x = 4$, the second derivative is positive and concave up. This corresponds to a relative minimum.

Practice

3. Use a second derivative to test whether each critical point of

$$f(x) = 2x^3 + 7.5x^2 + 9x + 7$$

is a relative maximum or relative minimum.

Question 3 – What is the point of diminishing returns?

Key Terms

Law of diminishing returns

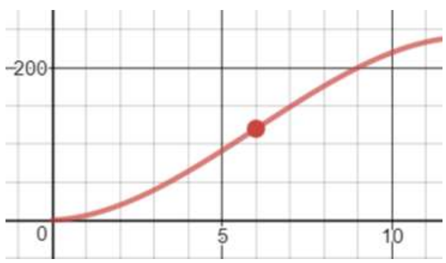
Point of diminishing returns

Summary

The Law of Diminishing Returns describes how an output changes as an input is increased.

> When an input is increased, such as labor, while all others are held constant, the resulting increase in production will become larger and larger until the point of diminishing returns. After the point of diminishing returns, the increase in production becomes smaller and smaller.

On a graph such as the one below, the rate (slope) increases from $x = 0$ to $x = 6$. This means the output is getting larger and larger.



However, after $x = 6$ the rate begins to decrease even though the graph continues to rise. This means the output is still getting larger, just at a smaller and smaller rate.

Since the point of diminishing returns is a point of inflection, it may be located by tracking the second derivative.

Notes

Guided Example

Find the point of diminishing returns for the function $R(x) = -0.25x^3 + 4.5x^2 + 2x$ where $R(x)$ represents revenue (in thousands of dollars) and x represents the amount spent on advertising (in thousands of dollars).

Solution A point of diminishing returns is an inflection point. To find it, we need to track the sign of the second derivative. Compute the derivatives:

$$R'(x) = -0.75x^2 + 9x + 2$$

$$R''(x) = -1.5x + 9$$

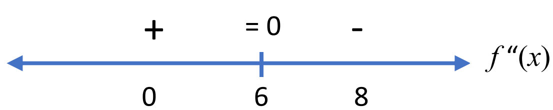
Set the second derivative equal to zero,

$$-1.5x + 9 = 0$$

$$-1.5x = -9$$

$$x = 6$$

Now make a number line for the second derivative and test on either side of $x = 6$.



Since the concavity changes at $x = 6$, it is a point of inflection. Substitute $x = 6$ into the revenue function to get the corresponding point:

$$R(6) = -0.25(6)^3 + 4.5(6)^2 + 2(6) = 120$$

This means the point of diminishing returns is located at (6, 120). When the company advertises up to \$6000, increasing advertising increases the revenue faster and faster. When the advertise beyond \$6000, the revenue still rises, but at a slower and slower rate.

Practice

1. Find the point of diminishing returns for the function $R(x) = -0.05x^3 + 6x^2 + 9x$ where $R(x)$ represents revenue (in thousands of dollars) and x represents the amount spent on advertising (in thousands of dollars).

Section 12.3 Optimizing Business Functions

Question 1 – How do you maximize the revenue or profit for a business?

Question 2 – How do you minimize the average cost for a business?

Question 1 – How do you maximize the revenue or profit for a business?

Key Terms

Revenue	Cost
Profit	Optimize

Summary

The process of finding a maximum or minimum of a function is called optimization. A good starting point for optimization in business and finance is the optimization of revenue, cost, and profit.

These functions may be given to you or may have to create them from data. You may also need to build these functions from other functions. For instance, revenue relates to demand via the equation

$$\text{revenue} = \text{quantity} \cdot \text{demand}$$

The revenue function may be a function of the quantity (like $R(x)$ or $R(Q)$) or it may be a function of price (like $R(p)$). You may be given a demand equation solve for quantity or a demand equation solved for price. In either case, the relationship above can be used to compute a revenue function.

The profit function is related to revenue and cost by the equation,

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

If we have revenue and cost functions, we can compute the profit by subtracting the revenue and cost.

Once we have a function, we can find its minimum or maximum using the strategies from Sections 12.1 and 12.2. For instance, if we are looking for and absolute extremum on a closed interval we need to examine the value of the function at the critical points and the endpoints. The highest value is the absolute maximum and the lowest value is the absolute minimum.

To find a relative extremum, we need to locate the critical points from the first derivative. Once this is accomplished, there are two strategies for deciding if the critical points are a relative maximum, relative minimum, or neither.

Strategy 1: First Derivative Test

Use a first derivative number line to track the sign of the first derivative.

- If the function changes from increasing to decreasing around the critical point it is a relative maximum.
- If the function changes from decreasing to increasing around the critical point it is a relative minimum.
- If the function slope does not change sign on either side of the critical point, it is not a relative extremum.

Strategy 2: Second Derivative Test

Compute the second derivative at the critical points.

- If the sign of the second derivative is negative at the critical point, the critical point is a relative maximum.
- If the sign of the second derivative is positive at the critical point, the critical point is a relative minimum.
- If the second derivative is zero at the critical point, we must use the First Derivative Test to classify the critical point.

<u>Notes</u>

Guided Example	Practice
The total profit $P(x)$ (in thousands of dollars) from the sale of x hundred units pf product is approximated by $$P(x) = -x^3 + 10x^2 + 9x - 90$$ a. Find the number of units that must be sold to maximize profit. **Solution** To maximize the profit function, we need to find the critical points of the function. Start with the derivative, $$P'(x) = -3x^2 + 20x + 9$$ and set it equal to zero: $$-3x^2 + 20x + 9 = 0$$ In previous examples, we factored the polynomial to solve it. However, for most examples in the real world this is not possible. To solve this quadratic equation, use the quadratic formula: $$x = \frac{-20 \pm \sqrt{20^2 - 4(-3)(9)}}{2(-3)} \approx -0.423, 7.09$$ Since x represents the number of products, the negative critical point is not a possible answer. To determine whether 7.09 is a maximum, minimum, or something else, we can construct a first derivative number line or evaluate the critical point in the second derivative. In this example, we choose the second strategy. The second derivative is $$P''(x) = -6x + 20$$ If we put the critical point into the second derivative, we get $$P''(7.09) = -6(7.09) + 20 = -22.54$$	1. The total profit $P(x)$ (in thousands of dollars) from the sale of x hundred units pf product is approximated by $$P(x) = -x^3 + 25x^2 + 100x - 2500$$ a. Find the number of units that must be sold to maximize profit.

Since the second derivative is negative, the graph is concave down at 7.09. This makes $x = 7.09$ a relative maximum. The maximum profit occurs at approximately 709 units. b. Find the maximum profit. **Solution** To find the corresponding profit, substitute x = 7.09 into the profit function, $P(7.09) = -(7.09)^3 + 10(7.09)^2 + 9(7.09) - 90$ ≈ 120.090 or \$120,090.	b. Find the maximum profit.

Guided Example | Practice

The demand function for a certain product is given by $p = 210 - \frac{x}{10}$ where x is the number of units sold (in thousands) and p is the price per unit (in dollars). a. Find the revenue function $R(x)$. **Solution** The revenue function is the product of quantity and price, $R(x) = x\,p$ Substitute the demand function in place of p to give $R(x) = x\left(210 - \frac{x}{10}\right)$ $= 210x - \frac{x^2}{10}$	2. The demand function for a certain product is given by $D(p) = 105 - 0.2p^2$ hundred units where p is the price per unit (in dollars). a. Find the revenue function $R(p)$.

b. Find the number of units sold that maximizes revenue.

Solution Take the derivative to find the critical points:

$$R'(x) = 210 - \frac{2x}{10}$$

Set this derivative equal to zero,

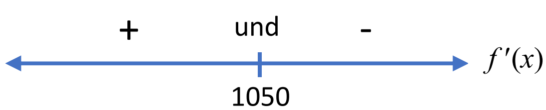
$$210 - \frac{2x}{10} = 0$$

$$2100 - 2x = 0$$

$$2100 = 2x$$

$$1050 = x$$

In this example, we'll determine if the critical point is a relative maximum or minimum using a first derivative number line:



Since the function changes from increasing to decreasing, the critical point must be a relative maximum. Revenue is maximized at 1,050,000 units

c. Find the maximum revenue.

Solution Substitute 1050 into the revenue function,

$$R(1050) = 210(1050) - \frac{(1050)^2}{10} = 110250$$

Since the quantity is in thousands and the price is in dollars, the revenue must be in thousands of dollars. At a production level of 1,050,000 the revenue is $110,250,000.

b. Find the price per unit that maximizes revenue.

c. Find the maximum revenue.

Guided Example	Practice
The demand function for a certain product is given by $p = -x + 20$ where x is the number of units sold (in hundreds) and p is the price per unit (in dollars). The cost to produce x hundred units is $C(x) = 4x + 10$ hundred dollars a. Find the revenue function $R(x)$. **Solution** The revenue function is the product of quantity and price, $R(x) = x\,p$ Substitute the demand function in place of p to give $R(x) = x(-x + 20)$ $= -x^2 + 20x$ Since the quantity is in hundreds and the price is in dollars, the revenue is in hundreds of dollars. b. Find the profit function $P(x)$. **Solution** Profit is the difference between revenue and cost, $P(x) = R(x) - C(x)$ Putting in the revenue and cost gives $P(x) = (-x^2 + 20x) - (4x + 10)$ $= -x^2 + 16x - 10$	3. The demand function for a certain product is given by $p = -0.5x + 10$ where x is the number of units sold (in hundreds) and p is the price per unit (in dollars). The cost to produce x hundred units is $C(x) = 2x + 8$ hundred dollars a. Find the revenue function $R(x)$. b. Find the profit function $P(x)$.

c. Find the number of units sold that maximizes profit. **Solution** The derivative of the profit function is $P'(x) = -2x + 16$ Setting this equal to zero yields $-2x + 16 = 0$ $-2x = -16$ $x = 8$ To determine if this critical point is a maximum, find the second derivative, $P''(x) = -2$ Since the second derivative is negative, the function is always concave down. This tells us that $x = 8$ is a relative maximum. d. Find the maximum profit. **Solution** Substitute $x = 8$ into the profit function, $P(8) = -(8)^2 + 16(8) - 10 = 54$ At a production level of 800 units, the profit is $5400.	c. Find the number of units sold that maximizes profit. d. Find the maximum profit.

Question 2 – How do you minimize the average cost for a business?

Key Terms

Cost Average Cost

Summary

The average cost function describes the cost per unit when a certain number of units are produced. It may be computed by dividing the cost by the quantity,

$$\overline{C}(x) = \frac{C(x)}{x}$$

where x is the quantity produced.

Typically, the average cost is minimized. We can compute the critical points and follow either strategy described earlier to find the absolute minimum or relative minimum.

Notes

Guided Example	Practice
The total cost to produce x thousand units of a product are $C(x) = 0.0025x^2 + 50$ thousand dollars a. Find the average cost function $\overline{C}(x)$. **Solution** The average cost function is defined to be $\overline{C}(x) = \frac{C(x)}{x}$ Put the cost function into the numerator to yield $\overline{C}(x) = \frac{0.0025x^2 + 50}{x}$ $= 0.0025x + 50x^{-1}$ b. Find the number of units that minimizes the average cost. **Solution** Find the critical points by finding the first derivative: $\overline{C}'(x) = 0.0025 - 50x^{-2}$ $= 0.0025 - \frac{50}{x^2}$ The derivative is undefined at $x = 0$ so this is a critical point. Other critical points come from setting the derivative equal to zero and solving for x, $0.0025 - \frac{50}{x^2} = 0$ $0.0025x^2 - 50 = 0$ $0.0025x^2 = 50$ $x^2 = \frac{50}{0.0025}$ $x = \pm\sqrt{\frac{50}{0.0025}} \approx \pm 141.421$	1. The total cost to produce x thousand units of a product are $C(x) = 0.005x^2 + 120$ thousand dollars a. Find the average cost function $\overline{C}(x)$. b. Find the number of units that minimizes the average cost.

To determine whether this corresponds to a relative minimum, compute the second derivative, $$\overline{C}''(x) = 100x^{-3}$$ $$= \frac{100}{x^3}$$ When we evaluate the positive critical point, we get $$\overline{C}''(141.421) = \frac{100}{(141.421)^3} > 0$$ Since the second derivative is positive, x = 141.421 is a relative minimum. So, 141,421 units minimizes average cost.	

Section 12.4 Business Applications of Extrema

Question 1 – How do you find the optimal dimensions of a product?

Question 2 – How do we make decisions about inventory?

Question 1 – How do you find the optimal dimensions of a product?

Key Terms

Optimization Objective function

Summary

In this question, we continue to optimize functions corresponding to applications. As in the previous section, we will maximize or minimize functions that model an application, However, in this section the functions to be maximized or minimized, called the objective function, will not be given. You will need to build the objective function based on the application given to you.

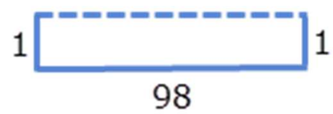
The applications may correspond to objects that needs to be built with a minimum amount of materials or with a maximum volume. They may also involve revenue, cost or profit functions with special circumstances such as discounts on the price. In each of these cases you will need a strategy to come up with the objective function. Often a table is useful to spot patterns in the information that you can use to make you objective function from. The guided examples below demonstrate this process of constructing tables to find the objective function.

Notes

Guided Example

Suppose you wish to construct a rectangular pet run along the side of your house using your house as one side of the enclosure. If you have 100 feet of fencing for the remaining three side of the pet run, what are the dimensions of the largest area that can be enclosed?

Solution The rectangular area has two dimensions, length and width. We'll think of the fence as being composed of a single length in one direction and two widths in the other direction. Let's construct one possibility to give us a sense of what we are trying to do.



If the widths of the pet run are both 1 foot, then there is 98 feet remaining for the length. This results in a pet run with area 1 · 98 or 98 square feet. Enter this in the first line of the table below.

Width	Length	Area
1	98	1 · 98 or 98
2	96	2 · 96 or 192
5	90	5 · 90 or 450
10	80	10 · 80 or 800
x	$100 - 2x$	$x(100 - 2x)$

Continue to enter possible widths in a systematic way trying widths like 2, 5, and 10. Then compute the corresponding length based on the fact that you have 100 feet of fence. Finally, write out the corresponding area for each set of dimensions.
To find the last row, assume the width is a variable, x. Based on the pattern in the second column, we know that the length must be the remaining fencing after the two width sections are subtracted. Then the area is the product of the length and the width. In terms of the width x, the objective function for the area is

$$A(x) = x(100 - 2x)$$

$$= 100x - 2x^2$$

To find the relative maximum of this function, find its critical points from the derivative:

$$A'(x) = 100 - 4x = 0$$

$$100 = 4x$$

$$25 = x$$

This critical number could be a relative maximum, a relative minimum, or neither. To classify the critical number, find the second derivative,

$$A''(x) = -4$$

Since the second derivative is always negative, the function is always concave down. This means any critical number is a relative maximum.

With a width of 25 feet, the length must be 50 feet giving a 25 by 50-foot enclosure with maximum area of $A(25) = 100(25) - 2(25)^2 = 1250$ square feet.

Practice

1. Suppose you wish to construct two adjacent rectangular pet runs along the side of your house using your house as one side of the enclosure. Each pet run is to be the same size and share the fencing where they are adjacent.

If you have 100 feet of fencing for the remaining sides of the pet run, what are the dimensions of the largest area that can be enclosed?

Guided Example

A local club is arranging a charter flight to London. The cost of the trip is \$500 each for 80 passengers, with a refund of \$10 per passenger for each passenger in excess of 80.

a. Find the revenue as a function of the number of passengers

Solution Since we are looking for a function of the number of passengers, let p represent the number of passengers. Assume that the number of passengers must be at least 80, so $p \geq 80$.

To create a table, we need to realize that the revenue comes from ticket sales to passengers, but we also need to consider the refund for each ticket.

Number of Passengers	Revenue from Tickets	Passengers in Excess	Refund per Passenger	Total Refund	Total Revenue
80	500(80)	0	0	0	500(80)
81	500(81)	1	5	5(81)	$500(81)-5(81)$
82	500(82)	2	10	10(82)	$500(82)-10(82)$
p	$500p$	$p-80$	$5(p-80)$	$5(p-80)p$	$500p-5(p-80)p$

Notice that as the number of passengers increases, the refund per passenger also increases.

The revenue is

$$\begin{aligned} R(p) &= 500p - 5(p-80)p \\ &= -5p^2 + 900p \end{aligned}$$

b. Find the number of passengers that will maximize the revenue received from the flight.

Solution To maximize the revenue function, we need to locate the critical points of $R(p)$. The derivative is

$$R'(p) = -10p + 900$$

Setting the derivative equal to zero yields

$$\begin{aligned} -10p + 900 &= 0 \\ -10p &= -900 \\ p &= 90 \end{aligned}$$

This critical point may be a relative minimum, relative maximum, or neither. To check, find the second derivative:

$$R''(p) = -10$$

Since the second derivative is always negative, the function is always concave down. This tells us the critical point at $p = 90$ is a relative maximum.

c. Find the maximum revenue.

Solution The revenue is maximized at $p = 90$. To find the maximum revenue, substitute this value into the revenue function $R(p) = -5p^2 + 900p$,

$$R(p) = -5(90)^2 + 900(90) = 40,500 \text{ dollars}$$

Practice

A local club is arranging a charter flight to Greece. The cost of the trip is $1000 each for 100 passengers, with a refund of $5 per passenger for each passenger in excess of 100.

a. Find the revenue as a function of the number of passengers

b. Find the number of passengers that will maximize the revenue received from the flight.

c. Find the maximum revenue.

Question 2 – How do we make decisions about inventory?

Key Terms

Economic lot size

Economic order quantity

Summary

Businesses keep inventory on hand to use or meet customer demand. However, they generally do not purchase inventory for the entire year since they will most likely need to store some of that inventory until it is needed. To lower the storage costs, businesses frequently make or order inventory throughout the year. This may lead to additional costs such as set up costs or ordering costs when the making or ordering the inventory also incurs a cost.

A company manufacturing a product in distinct lot sizes regularly throughout the year might have costs to set up production for each lot, costs to make the product, and costs to hold the product. If they manufacture inventory in a size that minimizes the total costs, they balance the storage costs with other costs. This size order is called the economic lot size and minimizes the total cost,

$$\text{Total Cost} = \text{Set Up Costs} + \text{Production Costs} + \text{Holding Costs}$$

A company that orders inventory regularly throughout the year faces a similar problem. In this context, their total costs consists of the cost of the product they are ordering, the holding cost, and the cost of reordering the product,

$$\text{Total Cost} = \text{Cost of the Product} + \text{Holding Costs} + \text{Ordering Costs}$$

The order size that minimizes the total cost is called the economic order quantity.

Notes

Guided Example

A manufacturing plant needs to make 500,000 LED bulbs annually. Each bulb costs \$15 to make and it costs \$10000 to set up the factory to produce the bulbs. It costs the plant \$1 to store a bulb for 1 year. How many bulbs should the plant produce in each batch to minimize their total costs?

Solution We need to start by finding the total cost as a function of the batch size. Start a table with the first column being the size of the batch. The last column will be the total cost. To connect the batch size to the total cost, break the total cost down into its component costs: set up costs, production costs, and holding costs. These form three columns in the middle of the table.

Start the table by producing all of the bulbs in one batch of 500,000. This means we will need to set up the factory once at a cost of 10,000. It will cost 15(500,000) to make the bulbs. With such a great number of bulbs produced all at once, we will need to store some throughout the year. At the beginning of the year we will have 500,000 and by the end of the year we will have 0. This means we will have $\frac{500,000+0}{2}$ or 250,000 in storage on average at a cost of \$1 each to store.

Size of Batch	Set Up Costs	Production Cost	Holding Cost	Total Cost
500,000	10,000(1)	15(500,000)	250,000(1)	7,760,000
250,000	10,000(2)	15(500,000)	125,000(1)	7,645,000
125,000	10,000(4)	15(500,000)	62,500(1)	7,602,500
500	10,000(1000)	15(500,000)	250(1)	17,500,250
Q	$10,000 \cdot \frac{500,000}{Q}$	15(500,000)	$\frac{Q}{2}(1)$	?

We continue with different batch sizes to see the pattern in each column. In the last row, assume a batch size of Q. With this batch size, we will need to set up the factory $\frac{500,000}{Q}$ times at a cost of \$10,000 each time. The production costs stay the same since we are still producing 500,000 bulbs. Finally, the average number in storage will be $\frac{Q}{2}$ at a cost of \$1 each. Adding these costs give a total cost of

$$C(Q) = 10,000 \cdot \frac{500,000}{Q} + 15(500,000) + \frac{Q}{2}(1)$$

To find the minimum, let's first simplify this function to make it easier to take the derivative:

$$C(Q) = 5,000,000,000Q^{-1} + 7,500,000 + \frac{Q}{2}$$

The derivative is

$$C'(Q) = -5{,}000{,}000{,}000Q^{-2} + \frac{1}{2}$$
$$= \frac{-5{,}000{,}000{,}000}{Q^2} + \frac{1}{2}$$

Set the derivative equal to zero and solve for Q:

$$\frac{-5{,}000{,}000{,}000}{Q^2} + \frac{1}{2} = 0$$

Multiply each term by $2Q^2$

$$-10{,}000{,}000{,}000 + Q^2 = 0$$
$$Q^2 = 10{,}000{,}000{,}000$$
$$Q = \sqrt{10{,}000{,}000{,}000}$$
$$Q = 100{,}000$$

To check to see if this is a relative minimum, take the second derivative,

$$C''(Q) = 10{,}000{,}000{,}000Q^{-3}$$
$$= \frac{10{,}000{,}000{,}000}{Q^3}$$

When the second derivative is evaluated at the critical point,

$$C''(100{,}000) = \frac{10{,}000{,}000{,}000}{100{,}000^3} > 0$$

We get a positive value. This means the cost function is concave up and the critical point is a relative minimum.

Practice

1. A manufacturing plant needs to make 200,000 LED bulbs annually. Each bulb costs \$8 to make and it costs \$1500 to set up the factory to produce the bulbs. It costs the plant \$0.50 to store a bulb for 1 year. How many bulbs should the plant produce in each batch to minimize their total costs?

Guided Example

The annual demand for a washing machines at a Best Buy store is 400. It costs \$0.50 to store a washer for one year. It costs \$15 to place an order for a washer. A washer costs \$300. How many washers should be ordered in each order to satisfy demand and to minimize cost?

Solution Following the same strategy as the previous guided example, create a table with order size in the first column and total cost in the last column. The middle columns are the different components of the total cost: ordering cost, storage cost, and the cost of the washers.

Order Size	Ordering Cost	Storage Cost	Cost of Washers	Total Cost
400	15(1)	200(0.50)	400(300)	120,115
200	15(2)	100(0.5)	400(300)	120,080
1	15(400)	0.5(0.5)	400(300)	126,000.25
Q	$15\left(\frac{400}{Q}\right)$	$\frac{Q}{2}(0.5)$	400(300)	?

Adding the costs from the bottom row gives us the total cost function,

$$C(Q) = 15\left(\frac{400}{Q}\right) + \frac{Q}{2}(0.5) + 400(300)$$

$$= 6000Q^{-1} + \frac{Q}{4} + 120,000$$

The derivative is

$$C'(Q) = -6000Q^{-2} + \frac{1}{4}$$

$$= \frac{-6000}{Q^2} + \frac{1}{4}$$

Set the derivative equal to zero and solve for Q:

$$\frac{-6000}{Q^2} + \frac{1}{4} = 0$$
$$-24,000 + Q^2 = 0$$
$$Q^2 = 24,000$$
$$Q = \sqrt{24,000}$$
$$Q \approx 154.9$$

Look at the second derivative to see if this is a relative minimum,

$$C''(Q) = 12,000Q^{-3}$$
$$= \frac{12,000}{Q^3}$$

Put in the critical point to give

$$C''(154.9) = \frac{12,000}{154.9^3} > 0$$

Since the second derivative is positive, the critical point is a relative minimum.

Practice

2. The annual demand for dryers at a Best Buy store is 500. It costs \$0.50 to store a dryer for one year. It costs \$10 to place an order for a dryer. A dryer costs \$250. How many dryers should be ordered in each order to satisfy demand and to minimize cost?

Chapter 12 Answers

Section 12.1

Question 1 1) a. Relative maximum at (-2, 1), relative minimum at (2,-1), b. No absolute maximum, absolute minimum at (2, -1).

2) a. Relative maximum at (-2, 0) and (1, 0), relative minimum at (-0.5, -1.2), b. Absolute maximum at (-2, 0) and (1, 0), no absolute minimum.

Question 2 1) Increasing (-2, 0) and (0, ∞), decreasing (-∞, -2)

2) Increasing (-∞, 1) and (1, ∞)

Question 3 1) Relative maximum at (-1, 21), relative minimum at (1.5, -10.25)

2) Relative maximum at (0, 0), relative minimum at (2, 4)

Question 4 1) Absolute maximum at (10, 3022), absolute minimum at (5, -103)

Section 12.2

Question 1 1) $\frac{d^2y}{dx^2} = -120x^3 - 2$, 2) $f''(x) = \frac{8}{(x-3)^2}$

Question 2 1) $\frac{d^3y}{dx^3} = -360x^2$, $\frac{d^4y}{dx^4} = -720x$

2) $f^{(3)}(x) = -24(x-3)^{-4}$, $f^{(4)}(x) = 96(x-3)^{-5}$

Question 3 1) Concave up (-∞, 1.667), Concave down (1.667, ∞)

2) Concave up (-1, ∞), Concave down (-∞, -1)

3) -3/2 is a relative maximum and -1 is a relative minimum

Question 4 1) Point of diminishing returns is (40, 6760)

Section 12.3

Question 1 1) a. 18.47 hundred units or 1847 units, b. 1547.650 thousand dollars or $1,547,650.

2) a. $R(p) = 105p - 0.2p^3$, b. $13.23, c. 926.01 hundred dollars or $92601

3) a. $R(x) = -0.5x^2 + 10x$, b. $P(x) = -0.5x^2 + 8x - 8$, c. 8 hundred units or 800, d. 24 hundred dollars or $2400.

Question 2 1) a. $\overline{C}(x) = \dfrac{0.005x^2 + 120}{x}$, b. 154.919 thousand units or 154,919 units

Section 12.4

Question 1 1) Width of 50/3 by a length of 50 to give an area of 2500/3

2) a. $R(p) = 1000p - 5(p-100)p$, b. 150 passengers, c. $112,500

Question 2 1) Approximately 34,641

2) Approximately 141