Section 13.1 Antiderivatives

Question 1 – What is an antiderivative?

Question 2 – What are the antiderivatives of some basic functions?

Question 3 - How do we find the antiderivative of functions that are combinations of basic functions?

Question 4 - How do we find the value of the arbitrary constant?

Question 1 – What is an antiderivative?

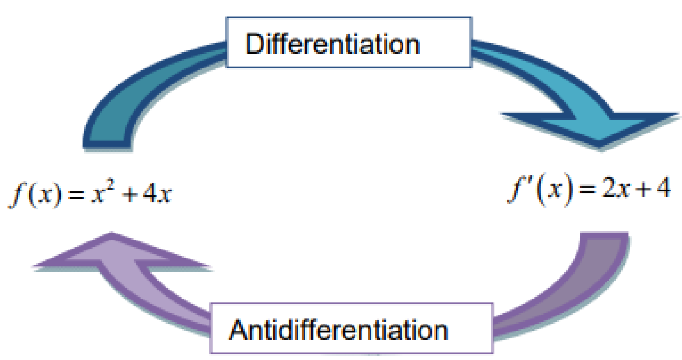
Key Terms

Antiderivative Antidifferentiation

Summary

In earlier chapters, we learned how to take the derivative of a function. We had many rules (especially the Power Rule for derivatives) that allowed us to take a function like $f(x)=x^2+4x$ and to find its derivative $f'(x)=2x+4$. This process, called differentiation, helps us to compute derivatives of functions.

In this question, we look at the reverse process called antidifferentiation. Now we start from the derivative and then compute the function the derivative came from. This function is called the antiderivative.



For the functions pictured above, we would say the antiderivative of $f'(x)=2x+4$ is $f(x)=x^2+4x$.

We can often deduce the antiderivative from our knowledge of derivative rules. In the case above, we know the derivative reduces the power by one. So the antiderivative must do the opposite or add one to the power. Knowledge of the derivative rules can often get us very close to the antiderivative. We can always fine tune our educated guesses by taking the derivative of the antiderivative.

When taking derivatives, we used the symbol $\frac{d}{dx}[\ ]$ to indicate that we want to take the derivative of the expression in brackets. For the example above, we would write

$$\frac{d}{dx}\left[x^2+4x\right]=2x+4$$

For antiderivatives, we use the symbol $\int(\ )\,dx$ to indicate the antiderivative of the function in parentheses. So, we could write

$$\int(2x+4)\,dx=x^2+4x+C$$

To indicate that the antiderivative of $2x+4$ is x^2+4x . An arbitrary constant *C* is added to the antiderivative because there are many antiderivatives of $2x+4$. Each antiderivative has a different constant symbolized by the *C*.

Notes

Guided Example

Find the antiderivative of $f'(x) = 3x^2 + 2x + 5$.

Solution Since you are given the derivative, you need to reverse the rules for derivatives. In this case, you are reversing the Power Rule

$$\frac{d}{dx}\left[x^n\right] = n\,x^{n-1}$$

Examine each term carefully and ask yourself, "What would you take the derivative of to get each term in $3x^2 + 2x + 5$?"

Based on your experience with derivatives, you would probably realize that the derivative of x^3 is $3x^2$ and that the derivative of x^2 is $2x$. But what about the last term?

Since the derivative of $5x$ is 5, we can deduce the antiderivative to be

$$f(x) = x^3 + x^2 + 5x + C$$

where C is some unknown constant.

Practice

1. Find the antiderivative of

$$f'(x) = 5x^4 + 4x^3 - 2$$

Guided Example

Find each integral

a. $\int 5x^4 z\, dz$

Solution The variable in this antiderivative is z. Because of this, we can ignore $5x^4$ and focus on the antiderivative of z. Rephrasing this, ask yourself, "What would you take the derivative of to get z?" Certainly, the power must be one higher so would z^2 work?

Close, but the derivative of z^2 is $2z$. So, try $\frac{1}{2}z^2$. Since $\frac{d}{dz}\left[\frac{1}{2}z^2\right] = z$, the antiderivative is

$$\int 5x^4 z\, dz = 5x^4 \cdot \tfrac{1}{2}z^2 + C$$

Practice

2. Find each integral

a. $\int 10 p^9 q\, dp$.

b. $\int 5x^4 z\ dx$ **Solution** In this antiderivative the variable is x. We can consider the z a constant and simply look for the antiderivative of $5x^4$. This antiderivative is x^5 . Putting this together, we get $$\int 5x^4 z\ dx = x^5 \cdot z + C$$	b. $\int 10 p^9 q\ dq$.

Notes

Question 2 – What are the antiderivatives of some basic functions?

Key Terms

Antiderivative

Summary

The rules for taking derivatives can be reversed to obtain the antiderivative rules. From Section 11.4, here are the derivative rules.

$\frac{d}{dx}\left[ax+b\right]=a$	$\frac{d}{dx}\left[x^n\right]=nx^{n-1}$	$\frac{d}{dx}\left[e^x\right]=e^x$
$\frac{d}{dx}\left[a^x\right]=(\ln a)a^x$	$\frac{d}{dx}\left[\log_a(x)\right]=\frac{1}{\ln(a)}\cdot\frac{1}{x}$	$\frac{d}{dx}\left[\ln(x)\right]=\frac{1}{x}$

The corresponding antiderivative rules are

$\int a\ dx = ax+C$	$\int x^n\ dx=\frac{x^{n+1}}{n+1}+C \quad \text{for} \quad n\neq -1$	$\int e^x\ dx = e^x+C$
$\int a^x\ dx=\frac{a^x}{\ln(a)}+C$	$\int \frac{1}{x}\ dx=\ln(x)+C \quad \text{for } x>0$	

Notes

Guided Example	Practice
Evaluate $\int x^{12}\, dx$ **Solution** Apply the Power Rule for Antiderivatives, $$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$$ with $n = 12$. This gives $$\int x^{12}\, dx = \frac{x^{13}}{13} + C$$	1. Evaluate $\int x^{20}\, dx$

Guided Example	Practice
Evaluate $\int \sqrt[3]{z}\, dz$ **Solution** To apply the Power Rule for Antiderivatives, rewrite the root $\sqrt[3]{z} = z^{1/3}$. This will give us, $$\int z^{1/3}\, dz = \frac{z^{1/3+1}}{1/3+1} + C$$ $$= \frac{z^{4/3}}{4/3} + C$$ $$= \frac{3}{4} z^{4/3} + C$$	2. Evaluate $\int \sqrt[4]{x}\, dx$

Guided Example	Practice
Evaluate $\int \frac{1}{t^2}\,dt$ **Solution** To apply the Power Rule for Antiderivatives, rewrite $\frac{1}{t^2} = t^{-2}$. This will give us, $$\begin{aligned}\int t^{-2}\,dt &= \frac{t^{-2+1}}{-2+1} + C \\ &= \frac{t^{-1}}{-1} + C \\ &= -\frac{1}{t} + C\end{aligned}$$	3. Evaluate $\int \frac{1}{u^3}\,du$

Guided Example	Practice
Evaluate $\int \frac{1}{z}\,dz$ **Solution** Like the last guided example, we might try to write $\frac{1}{z} = z^{-1}$. However, the Power Rule for Antiderivatives does not apply when n = -1. Instead we apply a different rule, $$\int \frac{1}{x}\,dx = \ln(x) + C \quad \text{for } x > 0$$ but with a different variable. Using z instead of x gives $$\int \frac{1}{z}\,dz = \ln(z) + C \quad \text{for } z > 0$$	4. Evaluate $\int 3^x\,dx$

Question 3 – How do we find the antiderivative of functions that are combinations of basic functions?

Key Terms

Antiderivative

Summary

Constants are all but ignored by the derivative. A similar property exists for antiderivatives that says for any real number constant *a*,

$$\int a\ f(x)\ dx = a\int f(x)\ dx$$

In effect, we can ignore the constant when taking an antiderivative and tack it on at the end.

Another property exists for breaking more complicated antiderivatives into smaller pieces. If you have a sum or difference of functions,

$$\int \left(f(x) \pm g(x) \right) dx = \int f(x)dx \pm \int g(x)dx$$

This means that we can break up sums and differences and compute the antiderivatives of the resulting pieces.

Notes

Guided Example	Practice
Evaluate $\int \frac{1}{2x}\,dx$ **Solution** The constant may be moved outside of the antiderivative and then we may take the antiderivative: $\int \frac{1}{2x}\,dx = \frac{1}{2}\int \frac{1}{x}dx$ $= \frac{1}{2}\ln(x) + C$ for $x > 0$	1. Evaluate $\int 5P^2\, dP$

Guided Example

Evaluate $\int \left(2x^4 - 5x^2 + 2x + 7\right) dx$

Solution Our rules for constants and sums or differences to break the polynomial into smaller pieces. The pieces can be evaluated with the Power Rule for Antiderivatives and the Constant Rule for Antiderivativess:

$$\int \left(2x^4 - 5x^2 + 2x + 7\right) dx = 2\int x^4 dx - 5\int x^2 dx + 2\int x\, dx + \int 7 dx$$

$$= 2\frac{x^5}{5} - 5\frac{x^3}{3} + 2\frac{x^2}{2} + 7x + C$$

$$= \frac{2}{5}x^5 - \frac{5}{3}x^3 + x^2 + 7x + C$$

Practice

2. Evaluate $\int \left(6x^3 - 7x^2 + 10x - 5\right) dx$

Guided Example

Evaluate $\int 10z\left(z^4 - 7z + 2\right) dz$

Solution Multiply the factors out and apply the Power Rule for Antiderivatives:

$$\begin{aligned}\int 10z\left(z^4 - 7z + 2\right) dz &= \int \left(10z^5 - 70z^2 + 20z\right) dz \\ &= 10\int z^5 dz - 70\int z^2 dz + 20\int z\, dz \\ &= 10\frac{z^6}{6} - 70\frac{z^3}{3} + 20\frac{z^2}{2} + C \\ &= \frac{5}{3}z^6 - \frac{70}{3}z^3 + 10z^2 + C\end{aligned}$$

Practice

3. Evaluate $\int \left(x^2 - 1\right)\left(x + 1\right) dx$

Guided Example	Practice
Evaluate $\int \frac{x^3 - 2x}{x}\,dx$ **Solution** Divide the *x* in the denominator into each term in the numerator and then apply antiderivative rules: $\int \frac{x^3 - 2x}{x}\,dx = \int\left(\frac{x^3}{x} - \frac{2x}{x}\right)dx$ $= \int\left(x^2 - 2\right)dx$ $= \int x^2 dx - \int 2\,dx$ $= \frac{x^3}{3} - 2x + C$	1. Evaluate $\int \frac{\sqrt{x} - 1}{x}\,dx$

Question 4 – How do we find the value of the arbitrary constant?

Key Terms

Antiderivative

Summary

All of our antiderivative rules for basic functions contain an arbitrary constant C. This is because when we take the derivative of constants, the result is zero. For instance,

$$\frac{d}{dx}\left[x^2+4x+1\right]=2x+4$$

$$\frac{d}{dx}\left[x^2+4x+10\right]=2x+4$$

$$\frac{d}{dx}\left[x^2+4x+99\right]=2x+4$$

Each derivative results in the same function. If we reverse the process, we can get many different antiderivatives. These antiderivatives differ by a constant. We would indicate this by writing

$$\int(2x+4)\,dx=x^2+4x+C$$

To single out a particular antiderivative, you need some information about it. Suppose I know that the antiderivative must pass through (2, 22). This means that when I put $x = 2$ into the antiderivative, the resulting y value should be $y = 22$:

$$2^2+4(2)+C=22$$

If we solve this for C we have the value of the constant:

$$12+C=22$$

$$C=10$$

So $f(x)=x^2+4x+10$ is the antiderivative of $f'(x)=2x+4$ that passes through (2, 22).

Notes

Guided Example	Practice
The derivative of a function $f(x)$ is $$f'(x) = 5x - 12$$ Find the function $f(x)$ that passes through (2, 7). **Solution** The antiderivative of this function is $$f(x) = \frac{5}{2}x^2 - 12x + C$$ To make sure the function passes through (2, 7), substitute $x = 2$ into $f(x)$ and set the result equal to 7: $$f(2) = \frac{5}{2}(2)^2 - 12(2) + C = 7$$ Solve this equation for C to give, $$\begin{aligned} \frac{5}{2}(2)^2 - 12(2) + C &= 7 \\ 10 - 24 + C &= 7 \\ -14 + C &= 7 \\ C &= 21 \end{aligned}$$ So, the function f (x) that passes through (2, 7) is $$f(x) = \frac{5}{2}x^2 - 12x + 21$$	1. The derivative of a function $f(x)$ is $$f'(x) = 2x - 6$$ Find the function $f(x)$ that passes through (5, 10).

Guided Example	Practice
Find the cost function if the marginal cost is given $$C'(x) = x^{1.1} + 5$$ and 5 units costs \$359.69. **Solution** To find the cost function, we need to compute the antiderivative of $C'(x)$, $$\begin{aligned} C(x) &= \int \left(x^{1.1} + 5 \right) dx \\ &= \int x^{1.1}\, dx + \int 5\, dx \\ &= \frac{x^{2.1}}{2.1} + 5x + C \end{aligned}$$ To make sure 5 units costs \$359.69, substitute x = 5 into the cost function and set the resulting expression equal to 359.69: $$C(5) = \frac{(5)^{2.1}}{2.1} + 5(5) + C = 359.69$$ Now solve the equation for C: $$\begin{aligned} \frac{(5)^{2.1}}{2.1} + 5(5) + C &= 359.69 \\ 13.984 + 25 + C &\approx 359.69 \\ 38.984 + C &\approx 359.69 \\ C &\approx 320.706 \end{aligned}$$ Put this constant into the antiderivative to yield $$C(x) = \frac{x^{2.1}}{2.1} + 5x + 320.706$$ Where the constant has been rounded to three decimal places.	2. Find the cost function if the marginal cost is given $$C'(x) = 2x + 25$$ and 10 units costs \$367.20.

Guided Example	Practice
Find the demand function $p(x)$ for the marginal revenue function $$R'(x) = 0.3x^2 - 0.4x + 112$$ Assume that if no items are sold, the revenue is 0. **Solution** Start by finding the revenue function from the marginal revenue function. Take the antiderivative of the marginal revenue to get $$\begin{aligned} R(x) &= \int \left(0.3x^2 - 0.4x + 112\right) dx \\ &= 0.1x^3 - 0.2x^2 + 112x + C \end{aligned}$$ We can find the value of C from the fact that revenue is zero when nothing is sold. This tells us that $R(0) = 0$. Put this information into the revenue function and solve for C: $$R(0) = 0.1(0)^3 - 0.2(0)^2 + 112(0) + C = 0$$ This leads to $C = 0$ and the resulting revenue function is $$R(x) = 0.1x^3 - 0.2x^2 + 112x$$ The demand function $p(x)$ is related to revenue $R(x)$ by the equation $$R(x) = x\ p(x)$$ Solving for p(x) gives $$p(x) = \frac{R(x)}{x}$$ Substitute the revenue into this equation to get $$\begin{aligned} p(x) &= \frac{0.1x^3 - 0.2x^2 + 112x}{x} \\ &= \frac{0.1x^3}{x} - \frac{0.2x^2}{x} + \frac{112x}{x} \\ &= 0.1x^2 - 0.2x + 112 \end{aligned}$$	3. Find the demand function $p(x)$ for the marginal revenue function $$R'(x) = 0.6x^2 - 0.8x + 127$$ Assume that if no items are sold, the revenue is 0.

Section 13.2 Approximating Area

Question 1 – Why is area important?

Question 2 – How is the area under a function approximated?

Question 1 – Why is area important?

<u>Key Terms</u>

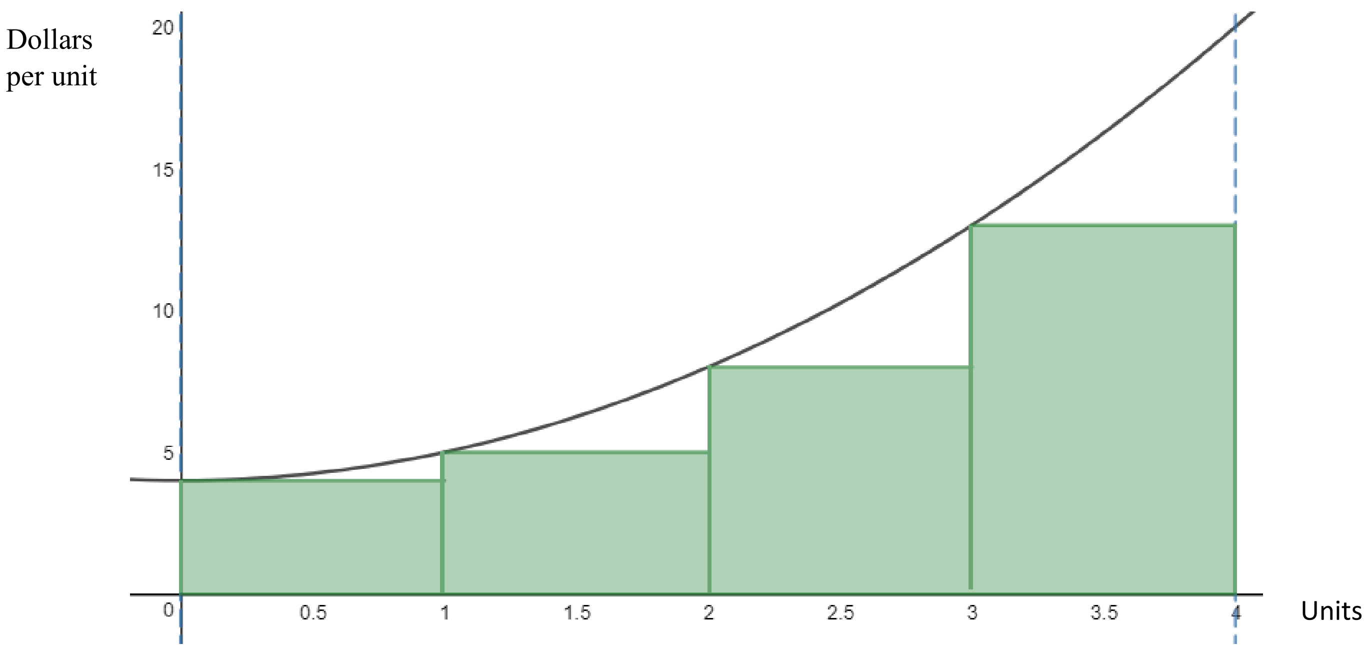
Area Right Hand Sum

Left Hand Sum

<u>Summary</u>

When a function is a rate such as marginal profit, the area under the function corresponds to a change in the profit function. We can approximate this change with estimates constructed from rectangles whose heights correspond to the rate.

Below is a marginal profit function $P'(x) = x^2 + 4$.



If the marginal profit is in dollars per unit, the area of each rectangle has units of

$$\left(\frac{\text{dollars}}{\cancel{\text{unit}}}\right)\left(\cancel{\text{units}}\right) = \text{dollars}$$

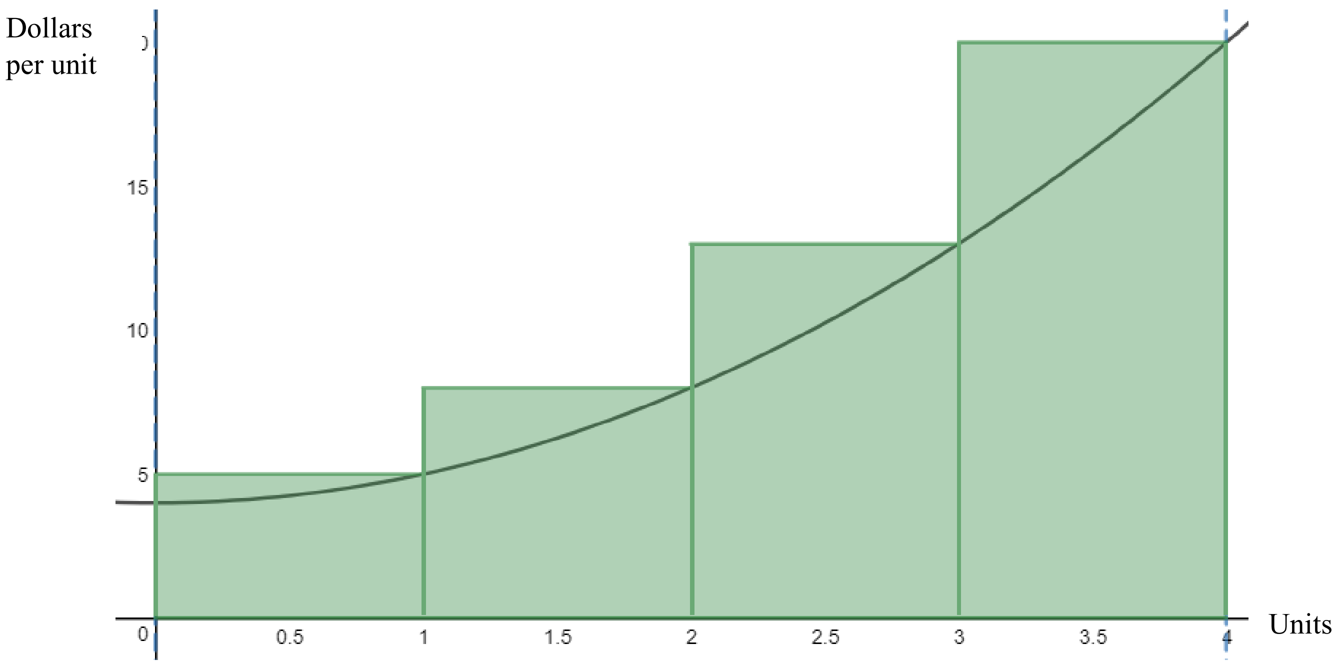
These are the units of profit. We can estimate the change in profit from x = 0 to x = 4 by summing the area of the rectangles:

$$\text{Estimate} = \left(4\frac{\text{dollars}}{\text{units}}\right)(1\text{ unit}) + \left(5\frac{\text{dollars}}{\text{units}}\right)(1\text{ unit}) + \left(8\frac{\text{dollars}}{\text{units}}\right)(1\text{ unit}) + \left(13\frac{\text{dollars}}{\text{units}}\right)(1\text{ unit})$$

$$= 30\text{ dollars}$$

Since each rectangle touches the graph on the left-hand side of the rectangle, this is called a left-hand estimate.

If the rectangles touch on the right-hand side of the rectangle, the estimate is called a right-hand estimate.



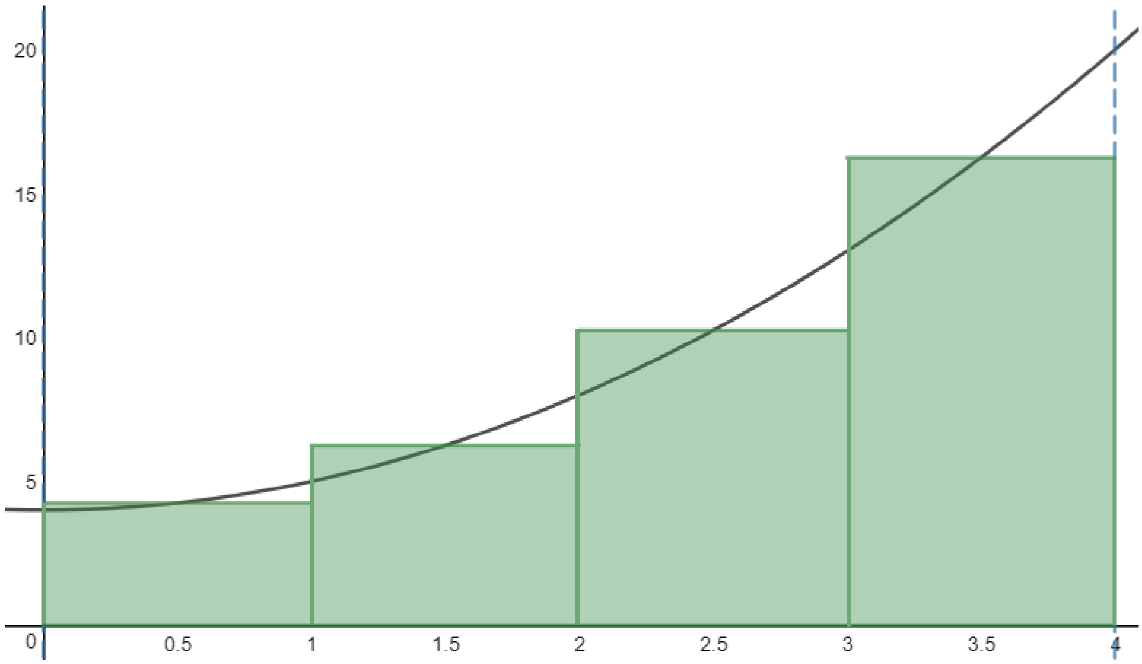
In this case the sum of the areas of the rectangles is

$$\text{Estimate} = \left(5\frac{\text{dollars}}{\text{units}}\right)(1\text{ unit}) + \left(8\frac{\text{dollars}}{\text{units}}\right)(1\text{ unit}) + \left(13\frac{\text{dollars}}{\text{units}}\right)(1\text{ unit}) + \left(20\frac{\text{dollars}}{\text{units}}\right)(1\text{ unit})$$

$$= 46\text{ dollars}$$

This estimate is an overestimate of the actual change in profit.

If the heights of the rectangles are found using the midpoint of each rectangle, the estimate is called a midpoint estimate.



The sum of the areas of these rectangles is

$$\text{Estimate} = \left(4.25\frac{\text{dollars}}{\text{units}}\right)(1\text{ unit}) + \left(6.25\frac{\text{dollars}}{\text{units}}\right)(1\text{ unit}) + \left(10.25\frac{\text{dollars}}{\text{units}}\right)(1\text{ unit}) + \left(16.25\frac{\text{dollars}}{\text{units}}\right)(1\text{ unit})$$

$$= 37\text{ dollars}$$

<u>Notes</u>

Guided Example

Use the table to find a lower and upper estimate of the change in profit when production is increased from 50 to 90 units.

Units	50	60	70	80	90
Rate of Change of Profit (Dollars per unit)	100	102	105	109	114

Solution Let's look at the interval from 50 to 60 units. The change in profit over this interval may be underestimated as

$$\text{Under} = \left(100\frac{\text{dollars}}{\text{unit}}\right)\left(10\text{ units}\right) = 1000\text{ dollars}$$

The overestimate is

$$\text{Over} = \left(102\frac{\text{dollars}}{\text{unit}}\right)\left(10\text{ units}\right) = 1020\text{ dollars}$$

The change in profit from 60 to 70 units may be underestimated as

$$\text{Under} = \left(102\frac{\text{dollars}}{\text{unit}}\right)\left(10\text{ units}\right) = 1020\text{ dollars}$$

Or overestimated as

$$\text{Over} = \left(105\frac{\text{dollars}}{\text{unit}}\right)\left(10\text{ units}\right) = 1050\text{ dollars}$$

Continue this process with similar products from 70 to 80 units as well as 80 to 90 units. The sum of the underestimates is

$$\text{Under} = \left(100\right)\left(10\right) + (102)(10) + (105)(10) + (109)(10) = 4160$$

This tells us that an underestimate of the change in profit from 50 to 90 units is \$4160.

Summing the overestimates gives

$$\text{Under} = \left(102\right)\left(10\right) + (105)(10) + (109)(10) + (114)(10) = 4300$$

This tells us that an overestimate of the change in profit from 50 to 90 units is \$4300.

Practice

1. Use the table to find a lower and upper estimate of the change in profit when production is increased from 10 to 50 units.

Units	10	20	30	40	50
Rate of Change of Profit (dollars per unit)	30	29	27	24	20

Question 2 – How is the area under a function approximated?

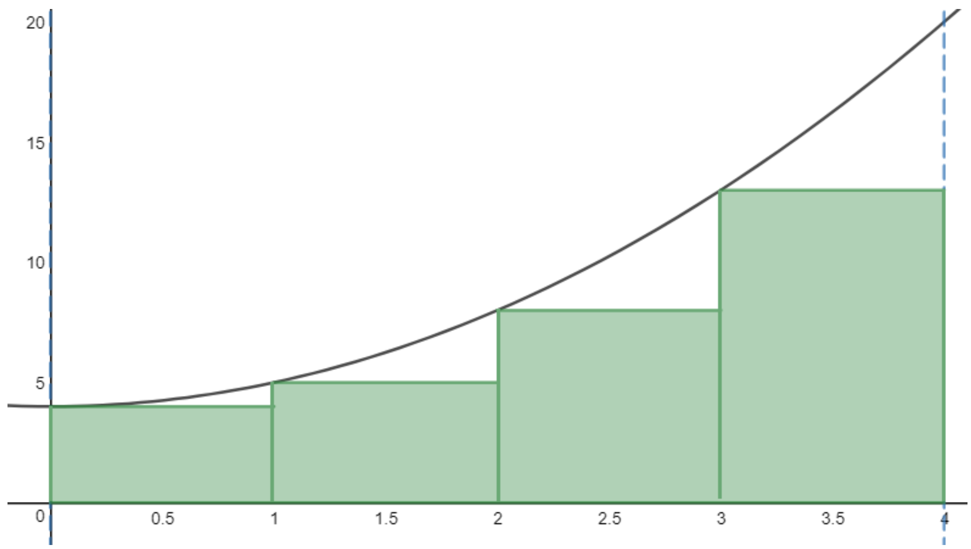
Key Terms

Left hand Sum Right hand sum

Summary

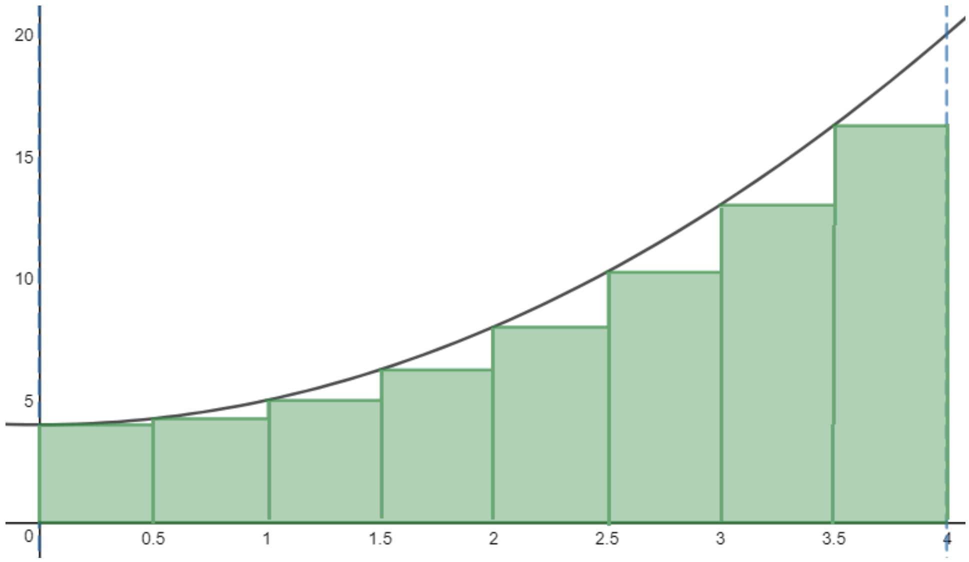
Left and right-hand sums approximate the area under a function's graph and above the x-axis with rectangles. The heights of each rectangle is determined by the function's y value on the left hand side of the rectangle or the right hand side of the rectangle.

Pictured below is a left-hand sum with 4 rectangles from $x = 0$ to $x = 4$.



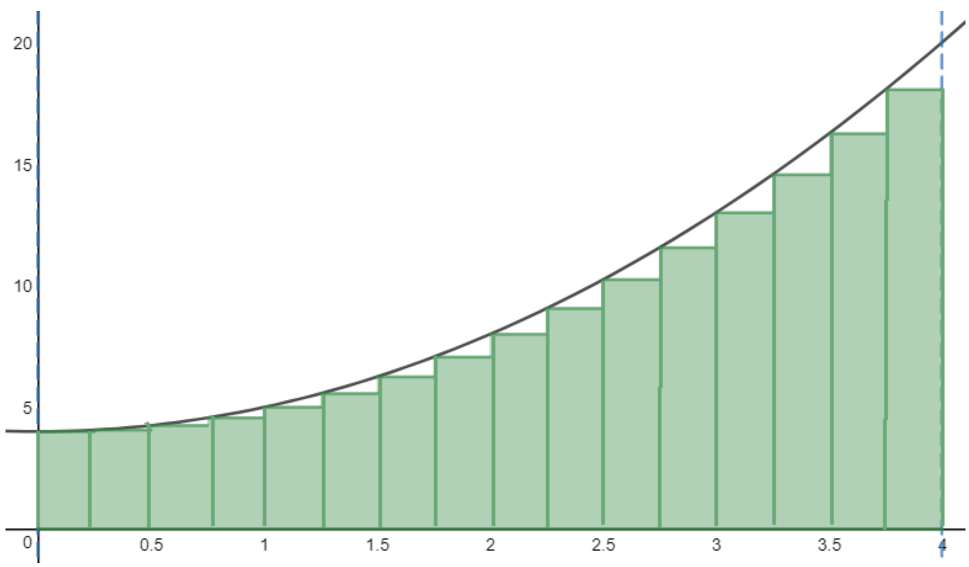
Each rectangle lies below the function's graph, so this estimate would result in an underestimate of the actual area under the function and above the x axis.

Compare this estimate with an estimate obtained from using 8 rectangles.



This estimate also results in an underestimate. But in this case the rectangles “fit” beneath the function resulting in an estimate that is closer to the exact area.

If we increase the number of rectangles to 16, we see that the rectangles fit even better with less of the actual area under the curve missed.



As the number of rectangles increases, the left-hand sum produces an underestimate that gets closer and closer to the exact area under the graph from $x = 0$ to $x = 4$.

A right-hand sum would produce an overestimate that get closer and closer to the exact area as the number of rectangles increases. This means the left and right-hand sums bracket the exact area and get closer and closer to the exact area as the number of rectangles increases.

<u>Notes</u>

Guided Example

Approximate the area under the graph of $f(x)$ and above the x-axis with rectangles from $x = 0$ to $x = 2$, using the following methods with $n = 4$.

$$f(x) = 9 - x^2$$

a. Use left endpoints.

Solution Since we need 4 rectangles from x = 0 to x = 2, each rectangle must be 0.5 wide. Since we are evaluating the function on the left side of the rectangle, the first rectangle is evaluated at $x = 0$. The left sum would be

$$\begin{aligned}\text{Left Sum} &= f(0)\cdot 0.5 + f(0.5)\cdot 0.5 + f(1)\cdot 0.5 + f(1.5)\cdot 0.5\\ &= 9\cdot 0.5 + 8.75\cdot 0.5 + 8\cdot 0.5 + 6.75\cdot 0.5\\ &= 16.25\end{aligned}$$

b. Use right endpoints.

Solution Since we are evaluating the function on the right side of the rectangle, the first rectangle is evaluated at $x = 0.5$. The right sum would be

$$\begin{aligned}\text{Right Sum} &= f(0.5)\cdot 0.5 + f(1)\cdot 0.5 + f(1.5)\cdot 0.5 + f(2)\cdot 0.5\\ &= 8.75\cdot 0.5 + 8\cdot 0.5 + 6.75\cdot 0.5 + 5\cdot 0.5\\ &= 14.25\end{aligned}$$

c. Average the answers in parts a and b.

Solution The average is $\dfrac{16.25 + 14.25}{2} = 15.25$

d. Use midpoints.

Solution Since we are evaluating the function in the middle of the rectangle, the first rectangle is evaluated at $x = 0.25$. The midpoint sum would be

$$\begin{aligned}\text{Midpoint Sum} &= f(0.25)\cdot 0.5 + f(0.75)\cdot 0.5 + f(1.25)\cdot 0.5 + f(1.75)\cdot 0.5\\ &= 8.9375\cdot 0.5 + 8.4375\cdot 0.5 + 7.4375\cdot 0.5 + 5.9375\cdot 0.5\\ &= 15.375\end{aligned}$$

Practice

1. Approximate the area under the graph of $f(x)$ and above the x-axis with rectangles from $x = 0$ to $x = 4$, using the following methods with $n = 4$.

$$f(x) = x^2 + 5$$

a. Use left endpoints.

b. Use right endpoints.

c. Average the answers in parts a and b.

d. Use midpoints.

Section 13.3 The Definite Integral

Question 1 – What is a definite integral?

Question 2 – How is the definite integral related to the approximate area?

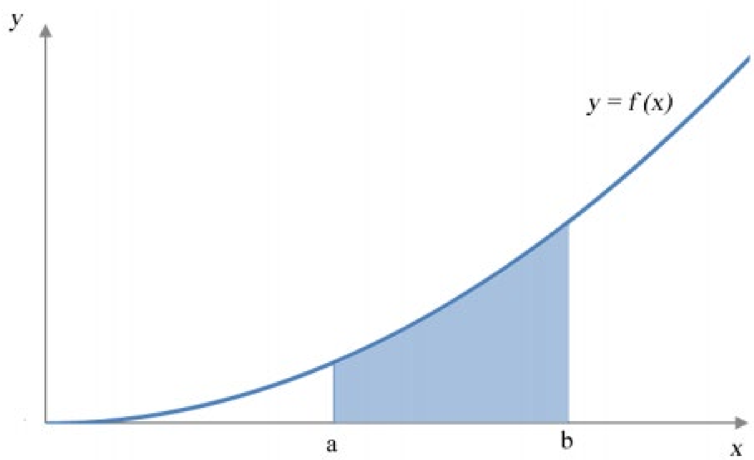
Question 1 – What is a definite integral?

Key Terms

Integral sign Definite integral

Summary

A definite integral is notation for indicating the area between a functions graph and the x axis from one x value to another x value.



The area of the shaded region above would be written as

$$\int_a^b f(x)\,dx$$

The numbers on either end of the integral indicate the x values we want to find the area between. The function between the integral and the dx is the graph we are finding the area under.

If a function's graph is above the *x* axis, the area between the curve and the *x* axis is positive. If the function's graph is below the axis, the area between the function and the *x* axis is negative.

Notes

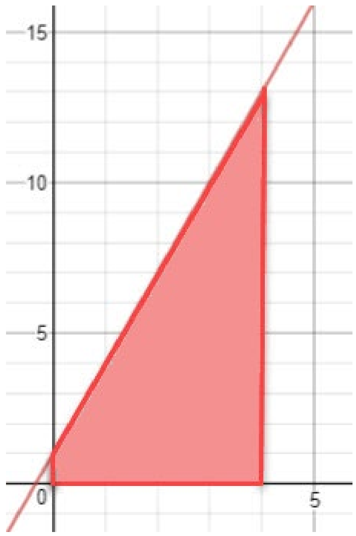
Guided Example

For the definite integral

$$\int_0^4 (3x+1)\,dx$$

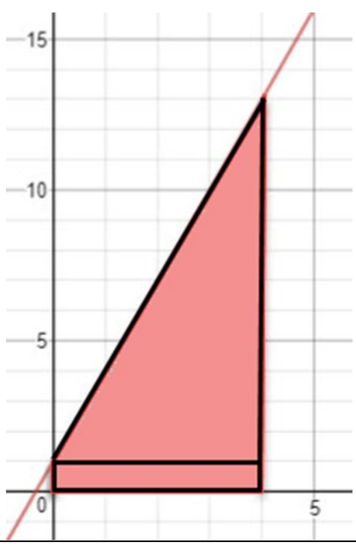
a. Graph the corresponding area.

Solution We are looking for the area from $x = 0$ to $x = 4$ under the curve $y = 3x + 1$.



b. Find the exact value of the area using a geometry formula.

Solution Break the area above into a triangle on top of a rectangle.



Practice

1. For the definite integral

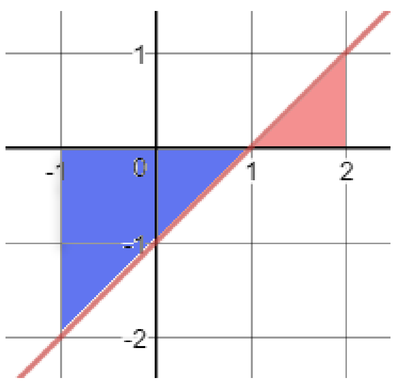
$$\int_0^3 (2x+2)\,dx$$

a. Graph the corresponding area.

b. Find the exact value of the area using a geometry formula.

The rectangle along the bottom has area 4 · 1. The triangle has area $\frac{1}{2}(4)(12)$. The shaded area is the sum of these areas or $$\int_0^4 (3x+1)\,dx = 4\cdot 1 + \frac{1}{2}(4)(12) = 28$$	

Guided Example | Practice

For the definite integral $$\int_{-1}^{2} (x-1)\,dx$$ a. Graph the corresponding area. **Solution** Graph $y = x - 1$ and shade the area between the graph and the x axis from $x = -1$ to $x = 2$.  Since part of this area is below the x axis, it is shaded in a different color than the area above the x axis. b. Find the exact value of the area using a geometry formula. **Solution** The darker area is below the x axis so it must be negative. The area of the darker triangle is $\frac{1}{2}(2)(2)$. The other triangle has area $\frac{1}{2}(1)(1)$. Putting this two together give the value of the definite integral,	2. For the definite integral $$\int_0^5 (-x+1)\,dx$$ a. Graph the corresponding area. b. Find the exact value of the area using a geometry formula.

$$\int_{-1}^{2}(x-1)\,dx=-2+\frac{1}{2}=-1.5$$ The area is negative because more of the shaded area is below the *x* axis than above the *x* axis.	

Question 2 – How is the definite integral related to the approximate area?

Key Terms

Riemann sum Summation

Summary

When we write out the sum of the areas of rectangles, we are computing Riemann sums. The rectangles have heights that are determined by a function. The widths are computed from the starting and ending x values of the region and the number of rectangles, we wish to put in the region.

The area of n rectangles under a function $f(x)$ is

$$f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

where Δx is the width of each rectangle. If the area starts at $x = a$ and ends at $x = b$, the width of each rectangle is

$$\Delta x = \frac{b-a}{n}$$

The values of each x value in the function is determined by whether you are computing the sums using the left side, right side, or midpoint of the rectangle.

This sum can be written more compactly using sigma notation as

$$\sum_{i=1}^{n} f(x_i)\,\Delta x$$

This sum approximates the area of a region under the graph and between two x values. If we use a larger and larger number of rectangles, the approximation becomes better and better. If we were to use an infinite number of rectangles, the area of the rectangles would match the area under the graph exactly. We express this by writing

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

Although we could never put an infinite number of rectangles under a function, we can symbolize it by writing the limit as n approaches infinity.

Notes

Guided Example

Let $f(x) = 2x + 5$ where $x_1 = 1$, $x_2 = 3$, $x_3 = 5$, $x_4 = 7$ and $\Delta x = 2$.

a. Compute $\sum_{i=1}^{4} f(x_i)\Delta x$.

Solution Start by putting in the numbers for the x_i and Δx :

$$\sum_{i=1}^{4} f(x_i)\Delta x = f(1)\cdot 2 + f(3)\cdot 2 + f(5)\cdot 2 + f(7)\cdot 2$$

Now compute the function values and simplify the resulting sum,

$$\sum_{i=1}^{4} f(x_i)\Delta x = 7\cdot 2 + 11\cdot 2 + 15\cdot 2 + 19\cdot 2 = 104$$

b. If the sum on part a corresponds to a left-hand sum, what definite integral is it approximating?

Solution Since the Riemann sum describes a left hand sum, x_1 must be the starting value for the definite integral. The height of the last rectangle comes from $x_4 = 7$. But this is the left side of the rectangle so it must extend to 9. This means the area under $f(x) = 2x + 5$ extends from 1 to 9 and the definite integral is

$$\int_1^9 (2x+5)\,dx$$

Practice

1. Let $f(x) = 5x - 1$ where $x_1 = 3$, $x_2 = 5$, $x_3 = 7$, $x_4 = 9$ and $\Delta x = 2$.

a. Compute $\sum_{i=1}^{4} f(x_i)\Delta x$.

b. If the sum on part a corresponds to a right-hand sum, what definite integral is it approximating?

Notes

Section 13.4 The Fundamental Theorem of Calculus

Question 1 – How do you calculate the total change of a quantity exactly?

Question 2 – What is the Fundamental Theorem of Calculus?

Question 1 – How do you calculate the total change of a quantity exactly?

Key Terms

Total Change

Summary

When we are given the rate of change of a function, we can compute the total change in that function by computing the antiderivative of the rate. For example, the antiderivative of the marginal revenue function (the rate of change of revenue) is the revenue function. Similarly, the antiderivative of the marginal profit function is the profit function.

Once we have computed the quantity from its rate of change, we can use it to compute the total change between two different levels by subtracting the quantity at those levels.

Notes

Guided Example	Practice
Assume the annual rate of change of debt for a country (in billions of dollars per year can be modeled by the function $$D'(t) = 100 + 64t + 2.1t^2$$ where t is the number of years after 2000. How much did the debt change between 2004 and 2006? **Solution** The function given is the rate of change of debt, $D'(t)$. We can compute the debt $D(t)$ by taking the antiderivative of this rate, $$\begin{aligned} D(t) &= \int D'(t)\,dt \\ &= \int \left(100 + 64t + 2.1t^2\right) dt \\ &= 100t + 32t^2 + 0.7t^3 + C \end{aligned}$$ We would normally try to find the value of C. However, when computing a change in the debt, the value of the constant is irrelevant. We want the debt between 2004 (at $t = 4$) and 2006 (at $t = 6$). Compute the debt at each of these levels, $$\begin{aligned} D(4) &= 100(4) + 32(4)^2 + 0.7(4)^3 + C \\ &= 956.8 + C \\ D(6) &= 100(6) + 32(6)^2 + 0.7(6)^3 + C \\ &= 1903.2 + C \end{aligned}$$ The total change is $$\begin{aligned} D(6) - D(4) &= (1903.2 + C) - (956.8 + C) \\ &= 946.4 \end{aligned}$$ Or 946.2 billion dollars. Notice that the C drops out meaning it does not need to be found to get the total change in debt.	1. Assume the annual rate of change of debt for a country (in billions of dollars per year can be modeled by the function $$D'(t) = 225 + 4.5t + 0.6t^2$$ where t is the number of years after 2000. How much did the debt change between 2001 and 2005?

Question 2 – What is the Fundamental Theorem of Calculus?

<u>Key Terms</u>

Fundamental Theorem of Calculus

<u>Summary</u>

The previous question suggests that we can compute the value of a definite integral by using the antiderivative. The Fundamental Theorem of Calculus is motivated by this idea.

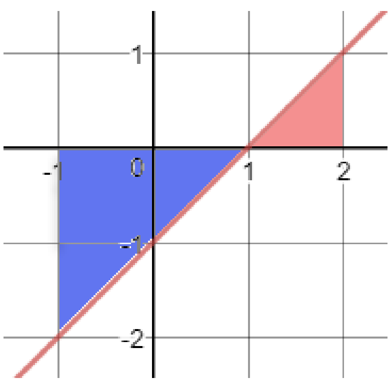
Fundamental Theorem of Calculus

Suppose *f* is a continuous function on the interval [*a*, *b*]. If *F* is an antiderivative of *f*, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

The means that the value of a definite integral may be computed by substituting the limits into the antiderivative and subtracting.

Let's look at an example. In a previous section, we computed found the value of a definite integral using geometry.



This area to the left corresponds to the definite integral $\int_{-1}^{2}(x-1)\,dx$. The sum of the area of the triangle below the x axis (equal to -2) and the area above the x axis (equal to ½). The sum of the shaded areas is -2 + ½ or -1 ½ .

Using the Fundamental Theorem of Calculus, we can compute the area using the antiderivative of $f(x) = x - 1$. The antiderivative is $F(x) = \frac{1}{2}x^2 - x + C$ so the definite integral is

$$\int_{-1}^{2}(x-1)\,dx = \left[\tfrac{1}{2}x^2 - x + C\right]_{-1}^{2}$$

The numbers on the right bracket indicate that they must be substituted into the antiderivative and the results subtracted

$$= \left(\tfrac{1}{2}(2)^2 - 2 + C\right) - \left(\tfrac{1}{2}(-1)^2 - (-1) + C\right)$$

$$= C - \left(\tfrac{3}{2} + C\right)$$

$$= -1.5$$

The solution with the Fundamental Theorem of Calculus gives the same value as the geometry solution. The advantage of the Fundamental Theorem of Calculus is that it can be applied even when the definite integral cannot be evaluated using geometry.

Notes

Guided Example

Evaluate $\int_0^4 \left(x^3 + x + 5\right) dx$

Solution The antiderivative of the integrand is $F(x) = \frac{1}{4}x^4 + \frac{1}{2}x^2 + 5x + C$. Applying the Fundamental Theorem of Calculus, we get

$$\begin{aligned}
\int_0^4 \left(x^3 + x + 5\right) dx &= \left[\tfrac{1}{4}x^4 + \tfrac{1}{2}x^2 + 5x + C\right]_0^4 \\
&= \left(\tfrac{1}{4}(4)^4 + \tfrac{1}{2}(4)^2 + 5(4) + C\right) - \left(\tfrac{1}{4}(0)^4 + \tfrac{1}{2}(0)^2 + 5(0) + C\right) \\
&= (92 + C) - C \\
&= 92
\end{aligned}$$

Practice

1. Evaluate $\int_{-1}^2 \left(2x^2 + 5x + 1\right) dx$

Guided Example

Evaluate $\int_1^3 \left(\frac{x^2 + x + 1}{x} \right) dx$

Solution To find the antiderivative of the integrand, divide the denominator into each term in the numerator:

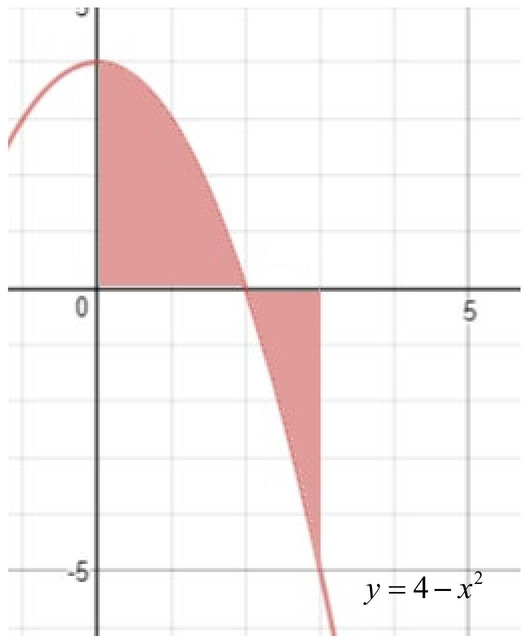
$$\begin{aligned}
\int_1^3 \left(\frac{x^2 + x + 1}{x} \right) dx &= \int_1^3 \left(\frac{x^2}{x} + \frac{x}{x} + \frac{1}{x} \right) dx \\
&= \int_1^3 \left(x + 1 + \frac{1}{x} \right) dx \\
&= \left[\tfrac{1}{2} x^2 + x + \ln(x) \right]_1^3 \\
&= \left(\tfrac{1}{2}(3)^2 + 3 + \ln(3) \right) - \left(\tfrac{1}{2}(1)^2 + 1 + \ln(1) \right) \\
&= 6 + \ln(3)
\end{aligned}$$

Practice

2. Evaluate $\int_1^4 \left(\frac{x + 2}{\sqrt{x}} \right) dx$

Guided Example

Find the total area of the shaded region (assume all areas are positive).



Solution To make sure the area below the x axis is positive, we calculate the area below the x axis separately:

$$\int_2^3 \left(4 - x^2\right) dx = \left[4x - \tfrac{1}{3}x^3 + C\right]_2^3$$

$$= \left(4(3) - \tfrac{1}{3}(3)^3 + C\right) - \left(4(2) - \tfrac{1}{3}(2)^3 + C\right)$$

$$= -\tfrac{7}{3}$$

The area above the x axis is

$$\int_0^2 \left(4 - x^2\right) dx = \left[4x - \tfrac{1}{3}x^3 + C\right]_0^2$$

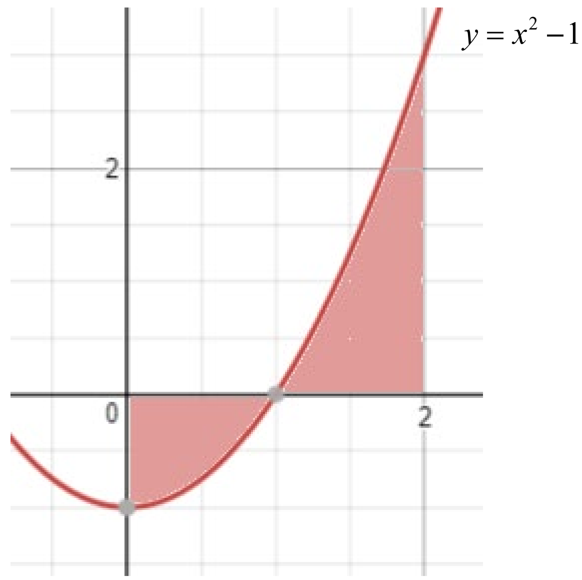
$$= \left(4(2) - \tfrac{1}{3}(2)^3 + C\right) - \left(4(0) - \tfrac{1}{3}(0)^3 + C\right)$$

$$= \tfrac{16}{3}$$

The total area is the sum of these individual areas, but with the opposite of the negative area, $\frac{7}{3} + \frac{16}{3} = \frac{23}{3}$.

Practice

3. Find the total area of the shaded region (assume all areas are positive).



Chapter 13 Answers

Section 13.1

Question 1 1) $f(x)=x^5-x^4-2x+C$, 2) a. $p^{10}q+C$, b. $10p^9\cdot\frac{1}{2}q^2+C$

Question 2 1) $\frac{x^{21}}{21}+C$, 2) $\frac{4}{5}x^{5/4}+C$, 3) $\frac{u^{-2}}{-2}+C$ or $-\frac{1}{2u^2}+C$, 4) $\frac{3^x}{\ln(3)}+C$

Question 3 1) $\frac{5}{3}P^3+C$, 2) $\frac{3}{2}x^4-\frac{7}{3}x^3+5x^2-5x+C$

3) $\frac{x^4}{4}+\frac{x^3}{3}-\frac{x^2}{2}-x+C$, 4) $2x^{1/2}-\ln(x)+C \quad x>0$

Question 4 1) $f(x)=x^2-6x+15$, 2) $C(x)=x^2+25x+17.20$,

3) $p(x)=0.2x^3-0.4x^2+127$

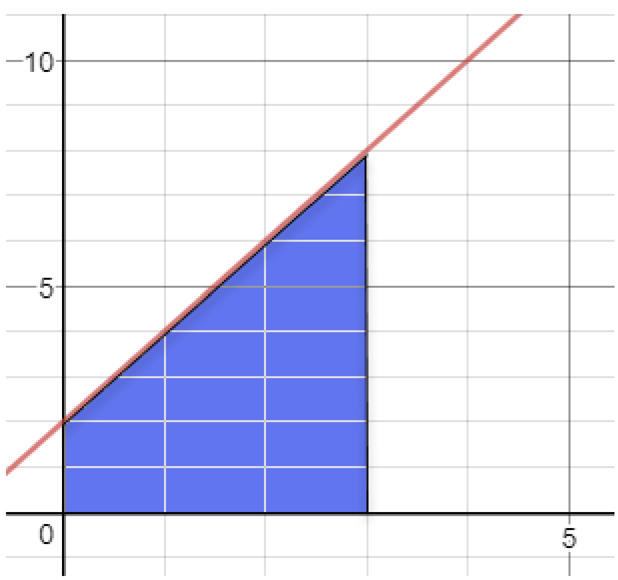
Section 13.2

Question 1 1) Underestimate is \$1000 and overestimate is \$1100.

Question 2 1) a. 34, b. 50, c. 42, d. 41

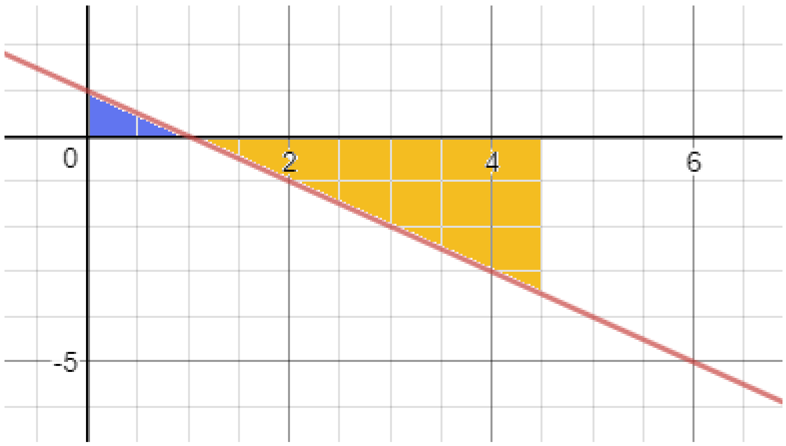
Section 13.3

Question 1 1) a.



b. 13.5

2) a.



b. -7.5

Question 2 1) a. 232, b. $\int_{1}^{9}(5x-1)\,dx$

Section 13.4

Question 1 1) 978.8

Question 2 1) 16.5

2) $\frac{26}{3}$

3) $\frac{2}{3}+\frac{4}{3}=2$