Section 2.4 Solving a System of Linear Equations with Matrices

Question 1 - What is a matrix?

Question 2 - How do you form an augmented matrix from a system of linear equations?

Question 3 - How do you use row operations to determine the reduced row echelon form of a matrix?

Question 4 - Do all systems of linear equations have unique solutions?

Question 5 - How do you mix different grades of ethanol to create a new grade of ethanol?

Question 1 - What is a matrix?

Key Terms

Matrix

Summary

A matrix is a table of numbers organized into rows and columns. Matrices are symbolized with capital letters. An example would be the matrix A below:

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 6 & 4 & 5 \end{bmatrix}$$

The entries in a matrix are placed within brackets. The size of a matrix is described by the number of rows and columns in the matrix. The matrix A above is a 2 × 3 matrix (read 2 by 3) since it has 2 rows and 3 columns. When quoting the size of a matrix, the number of rows are listed first and the number of columns is listed second.

We can refer to the entries using lower case letters and subscripts. The notation a_{mn} is the entry in the matrix A in the mth row and nth column. In the matrix above, $a_{21} = 6$ is the entry in the second row and third column.

Notes

Guided Example	Practice
Suppose you are given the matrix $$A = \begin{bmatrix} 1 & -2 \\ 3 & 0 \\ -1 & 5 \end{bmatrix}$$	1. Suppose you are given the matrix $$B = \begin{bmatrix} -1 & 3 & -2 \\ 2 & 1 & -3 \\ 0 & 4 & 5 \end{bmatrix}$$
a. Find the size of the matrix. **Solution** The matrix has three rows and two columns, so it is a 3 × 2 matrix.	a. Find the size of the matrix.
b. Find the value of a_{12} . **Solution** The entry a_{12} is the entry in the first row and second column so $a_{12} = -2$.	b. Find the value of b_{23} .
c. Find the value of a_{32} . **Solution** The entry a_{32} is the entry in the third row and second column so $a_{12} = 5$.	c. Find the value of b_{31} .

Question 2 - How do you form an augmented matrix from a system of linear equations?

Key Terms

Augmented matrix

Summary

An augmented matrix is a matrix whose entries correspond to a system of linear equations. For instance, if we have the system of linear equations,

$$\begin{aligned} x+2y+3z&=10 \\ -x+4y-2z&=3 \\ 2y+z&=5 \end{aligned}$$

the augmented matrix corresponding to the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ -1 & 4 & -2 & 3 \\ 0 & 2 & 1 & 5 \end{array}\right]$$

Comparing the system with the matrix. We see that the fourth column corresponds to the constants on the right side of the equal signs in the system. The vertical line corresponds to the equal signs in the system. Each column in the matrix matches the coefficients on the variables in the system. Also note that all terms with variable are on one side of the equation and the constants are on the opposite side.

Notes

Guided Example	Practice
Write an augmented matrix for each of the system of equations below.	1. Write an augmented matrix for each of the system of equations below.
a. $\begin{aligned} 2x+3y&=9 \\ x-\ \ y&=-1 \end{aligned}$	a. $\begin{aligned} -5x+2y&=12 \\ 3x-4y&=1 \end{aligned}$
Solution The variable terms are on one side of the equal sign and the constants are on the other side. Using the coefficients and the constants gives us the augmented matrix $\left[\begin{array}{cc\|c} 2 & 3 & 9 \\ 1 & -1 & -1 \end{array}\right]$	
b. $\begin{aligned} 2x+4y-\ \ z&=0 \\ x\qquad -4z&=9 \\ y+2z&=6 \end{aligned}$	b. $\begin{aligned} x\qquad -2z&=7 \\ 3x-y-4z&=-1 \\ 11x+y+2z&=2 \end{aligned}$
Solution Since this system has more variables and more equations, it results in a 3 × 4 augmented matrix. $\left[\begin{array}{ccc\|c} 2 & 4 & -1 & 0 \\ 1 & 0 & -4 & 9 \\ 0 & 1 & 2 & 6 \end{array}\right]$	
c. $\begin{aligned} y&=2x+1 \\ x+y&=7 \end{aligned}$	c. $\begin{aligned} x+3&=-5y+1 \\ 2x-y&=6 \end{aligned}$
Solution The variables are not on the same side of the equation. To fix this, subtract $2x$ from both sides of the first equation to give the system $\begin{aligned} -2x+y&=1 \\ x+y&=7 \end{aligned}$ The corresponding augmented matrix is $\left[\begin{array}{cc\|c} -2 & 1 & 1 \\ 1 & 1 & 7 \end{array}\right]$	

Guided Example	Practice
For each of the augmented matrices below, write the corresponding system of equations. a. $\left[\begin{array}{cc\|c} 2 & -1 & 5 \\ 3 & 1 & 4 \end{array}\right]$ **Solution** Assume that the first and second columns correspond t the variables x and y respectively. The corresponding system of linear equations is $$\begin{aligned} 2x - y &= 5 \\ 3x + y &= 4 \end{aligned}$$ b. $\left[\begin{array}{ccc\|c} 1 & 0 & 3 & 2 \\ 0 & 3 & 5 & 2 \\ 0 & 2 & -7 & 10 \end{array}\right]$ **Solution** Assume that the first three columns correspond to x, y, and z. The corresponding system of linear equations is $$\begin{aligned} x \quad + 3z &= 2 \\ 3y + 5z &= 2 \\ 2y - 7z &= 10 \end{aligned}$$	2. For each of the augmented matrices below, write the corresponding system of equations. a. $\left[\begin{array}{cc\|c} -3 & 9 & 1 \\ 2 & -5 & 3 \end{array}\right]$ b. $\left[\begin{array}{ccc\|c} 1 & 2 & -1 & 7 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 3 & 9 \end{array}\right]$

Guided Example	Practice
Using the variables *x*, *y* and *z*, write the solution corresponding to the augmented matrix $$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$ **Solution** The corresponding system of linear equations is $$x=4$$ $$y=2$$ $$z=-1$$	3. Using the variables *x*, *y* and *z*, write the solution corresponding to the augmented matrix $$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Question 3 - How do you use row operations to determine the reduced row echelon form of a matrix?

Key Terms

Reduced row echelon form Equivalent matrix

Summary

To solve a system of linear equations using matrices, we follow a strategy

Strategy for Solving a System of Linear Equations with Matrices

1. Determine the augmented matrix that corresponds to the given system of equations.
2. Find the reduced row echelon form of the augmented matrix.
3. Use the reduced row echelon form to obtain the solution of the original system of equations.

A system is in reduced row echelon form if the leading nonzero entry in each row is a one and all entries above and below the leading ones are zeros. To get the matrix into reduced row echelon form, we need to carry out row operations on the augmented matrix. Row operations yield augmented matrices that are equivalent. In other words, they correspond to systems that have the same solutions.

Row Operations on Matrices

Each of the row operations below changes a matrix to an equivalent matrix:

1. Interchange any two rows of a matrix.

2. Multiply a row by a nonzero constant.

3. Replace a row with the sum of a nonzero multiple of one row to a nonzero multiple of another row.

The process of converting an augmented matrix to reduced row echelon form is called Gauss-Jordan Elimination.

Let's look at how we would carry out row operations to solve the system of linear equations,

$$\begin{aligned} x+2y-5z&=-18 \\ 3x+5y+13z&=32 \\ -2x-2y+8z&=26 \end{aligned}$$

Operation	Arithmetic	Corresponding Matrix
		$\left[\begin{array}{ccc\|c} 1 & 2 & -5 & -18 \\ 3 & 5 & 13 & 32 \\ -2 & -2 & 8 & 26 \end{array}\right]$
$-3R_1+R_2\to R_2$ $2R_1+R_3\to R_3$	$\begin{array}{cccc} -3 & -6 & 15 & 54 \\ 3 & 5 & 13 & 32 \\ \hline 0 & -1 & 28 & 86 \end{array}$ $\begin{array}{cccc} 2 & 4 & -10 & -36 \\ -2 & -2 & 8 & 26 \\ \hline 0 & 2 & -2 & -10 \end{array}$	$\left[\begin{array}{ccc\|c} 1 & 2 & -5 & -18 \\ 0 & -1 & 28 & 86 \\ 0 & 2 & -2 & -10 \end{array}\right]$
$-1R_2\to R_2$	$\begin{array}{cccc} 0 & -1 & 28 & 86 \\ & & \times & -1 \\ \hline 0 & 1 & -28 & -86 \end{array}$	$\left[\begin{array}{ccc\|c} 1 & 2 & -5 & -18 \\ 0 & 1 & -28 & -86 \\ 0 & 2 & -2 & -10 \end{array}\right]$
$-2R_2+R_1\to R_1$	$\begin{array}{cccc} 0 & -2 & 56 & 172 \\ 1 & 2 & -5 & -18 \\ \hline 1 & 0 & 51 & 154 \end{array}$	$\left[\begin{array}{ccc\|c} 1 & 0 & 51 & 154 \\ 0 & 1 & -28 & -86 \\ 0 & 0 & 54 & 162 \end{array}\right]$

$-2R_2 + R_3 \to R_3$	$\begin{array}{cccc} 0 & -2 & 56 & 172 \\ 0 & 2 & -2 & -10 \\ \hline 0 & 0 & 54 & 162 \end{array}$	
$\frac{1}{54}R_3 \to R_3$	$\begin{array}{cccc} 0 & 0 & 54 & 162 \\ & & & \times\frac{1}{54} \\ \hline 0 & 0 & 1 & 3 \end{array}$	$\left[\begin{array}{ccc:c} 1 & 0 & 51 & 154 \\ 0 & 1 & -28 & -86 \\ 0 & 0 & 1 & 3 \end{array}\right]$
$-51R_2 + R_1 \to R_1$ $28R_2 + R_3 \to R_3$	$\begin{array}{cccc} 0 & 0 & -51 & -153 \\ 1 & 0 & 51 & 154 \\ \hline 1 & 0 & 0 & 1 \end{array}$ $\begin{array}{cccc} 0 & 0 & 28 & 84 \\ 0 & 1 & -28 & -86 \\ \hline 0 & 1 & 0 & -2 \end{array}$	$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

This gives the solution, $x = 1$, $y = -2$, and $z = 3$.

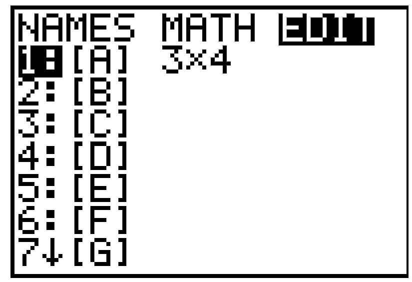
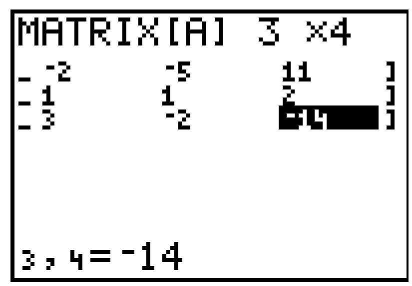
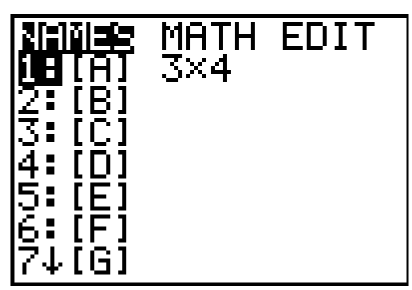
A graphing calculator also contains a command that will put a matrix into reduced row echelon form. For example, suppose we want to utilize your graphing calculator to solve the system

$$\begin{aligned} 4x - 2y - 5z &= 11 \\ x + y + z &= 2 \\ -2x + 3y - 2z &= -14 \end{aligned}$$

Start by converting this system to an augmented matrix,

$$\left[\begin{array}{ccc:c} 4 & -2 & -5 & 11 \\ 1 & 1 & 1 & 2 \\ -2 & 3 & -2 & -14 \end{array}\right]$$

Your calculator can put a matrix into reduced row echelon form using the rref command.

Enter the Matrix 1. Press [2nd] [x^{-1}] to access the MATRIX menu. 2. Use [▶] to go to EDIT. 3. Press [1] or move the cursor to 1: [A] and press [ENTER]. Note that if you used this matrix name before, it will have a dimension next to it.	
4. Enter the dimension of matrix A as 3 x 4. 5. Enter the values into the matrix as shown. Press [ENTER] after each entry. Note that the position is given at the bottom of the screen as 3,1=1 etc. This matrix will need two screens. Use [▶] to see last column and to enter. 6. Press [2nd] [MODE] to QUIT and return to the home screen.	
View the Matrix on the Home Screen 7. Press [2nd] [x^{-1}] to access the MATRIX menu. You are in the NAMES menu.	

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8. Move the cursor to 1: [A] and press ENTER. This will put [A] on the Home screen.
9. Press ENTER to view the matrix on the home screen. You may need to use the right arrow to scroll through the entire matrix.
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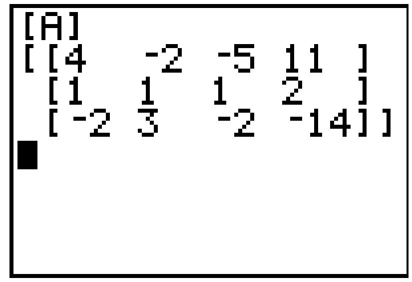

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<u>Notes</u>

Guided Example	Practice
Carry out the row operation: Replace R_1 with $\frac{1}{2}R_1$. $$\left[\begin{array}{ccc\|c} 2 & 4 & -10 & 2 \\ 4 & 2 & -3 & 3 \\ 1 & -3 & 4 & 2 \end{array}\right]$$ **Solution** Multiply each entry in the first row by $\frac{1}{2}$: $$\begin{array}{ccc\|c} 2 & 4 & -10 & 2 \\ & \times & & \frac{1}{2} \\ \hline 1 & 2 & -5 & 1 \end{array}$$ And replace the first row with this product, $$\left[\begin{array}{ccc\|c} 1 & 2 & -5 & 1 \\ 4 & 2 & -3 & 3 \\ 1 & -3 & 4 & 2 \end{array}\right]$$	1. Carry out the row operation: Replace R_2 with $\frac{1}{4}R_2$. $$\left[\begin{array}{ccc\|c} 2 & 4 & -10 & 2 \\ 4 & 2 & -3 & 3 \\ 1 & -3 & 4 & 2 \end{array}\right]$$

Guided Example	Practice
Carry out the row operation: Replace R_3 with $-3R_1 + R_3$. $$\left[\begin{array}{ccc\|c} 1 & 3 & -1 & 1 \\ 0 & 2 & 2 & -1 \\ 3 & 1 & 2 & -1 \end{array}\right]$$ **Solution** Start by working out the row operation $$\begin{array}{r\|ccc\|c} -3R_1: & -3 & -9 & 3 & -3 \\ +R_3: & 3 & 1 & 2 & -1 \\ \hline & 0 & -8 & 5 & -4 \end{array}$$	2. Carry out the row operation. a. Replace R_2 with $-2R_1 + R_2$. $$\left[\begin{array}{ccc\|c} 1 & 0 & -1 & 2 \\ 2 & 4 & 2 & 10 \\ 5 & 4 & 2 & 1 \end{array}\right]$$

And then replace the third row with this new row $\left[\begin{array}{ccc\|c} 1 & 3 & -1 & 1 \\ 0 & 2 & 2 & -1 \\ 0 & -8 & 5 & -4 \end{array}\right]$	b. Starting from your answer for part a, replace R_3 with $-5R_1 + R_3$.

Guided Example

Practice

Solve the system of equations below with Gauss-Jordan elimination. $\begin{aligned} x + \quad y &= 10 \\ 0.08x + 0.12y &= 0.84 \end{aligned}$ **Solution** Rewrite the system in an augmented matrix to get $\left[\begin{array}{cc\|c} 1 & 1 & 10 \\ .08 & .12 & .84 \end{array}\right]$ The leading entry in the first row is already a 1, so change the first entry, second row to a zero:	3. Solve the system of equations below with Gauss-Jordan elimination. $\begin{aligned} 2x + 5y &= -6 \\ -3x + y &= -8 \end{aligned}$

$-0.08R_1 + R_2 \to R_2 \quad \left[\begin{array}{cc:c} 1 & 1 & 10 \\ 0 & .04 & .04 \end{array}\right]$ To put a 1 in place of the leading entry in the second row: $\frac{1}{.04}R_2 \to R_2 \quad \left[\begin{array}{cc:c} 1 & 1 & 10 \\ 0 & 1 & 1 \end{array}\right]$ Finally, change the 1 in the first row, second column to a zero: $-1R_2 + R_1 \to R_1 \quad \left[\begin{array}{cc:c} 1 & 0 & 9 \\ 0 & 1 & 1 \end{array}\right]$ This gives the solution $(x, y) = (9,1)$.	

Question 4 - Do all systems of linear equations have unique solutions?

Key Terms

Summary

We can use Gauss-Jordan Elimination to solve a system when it does not have a solution or has many solutions. Start by carrying out the process of placing 1's and 0's in the matrix using row operations. Once the matrix is in reduced row echelon form, convert the matrix back to an equation. Once we have the equations, you would follow the same procedure that we used in Section 2.3 for inconsistent and dependent systems.

For instance, suppose a system of linear equations results in the reduced row echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 10 \\ 0 & 1 & 2 & 12 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

The last row of the matrix converts to the equation 0 = 1. Since this is a contradiction, the original system of equations has no solutions.

Suppose a system of linear equations results in the reduced row echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last row of the augmented matrix corresponds to the identity 0 = 0. In this situation, we rewrite the first two rows of the matrix as equations,

$$x + z = 5$$
$$y - z = 6$$

And solve for x and y to get

$$x = -z + 5$$
$$y = z + 6$$

This corresponds to the ordered triple $(-z+5, z+6, z)$.

Notes

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Solve the system of equations below with Gauss-Jordan elimination.

$$\begin{aligned} x + \quad y + \quad z &= 120 \\ 0.04x + 0.06y + 0.1z &= 6 \end{aligned}$$

Solution The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 120 \\ 0.04 & 0.06 & 0.1 & 6 \end{array}\right]$$

Follow the steps to put the augmented matrix in reduced row echelon form:

$$-0.04R_1 + R_2 \rightarrow R_2 \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 120 \\ 0 & 0.02 & 0.06 & 1.2 \end{array}\right]$$

$$\tfrac{1}{0.02}R_2 \rightarrow R_2 \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 120 \\ 0 & 1 & 3 & 60 \end{array}\right]$$

$$-1R_2 + R_1 \rightarrow R_1 \quad \left[\begin{array}{ccc|c} 1 & 0 & -2 & 60 \\ 0 & 1 & 3 & 60 \end{array}\right]$$

The augmented matrix is now in reduced row echelon form. Converting it back to equations gives

$$\begin{aligned} x - 2y &= 60 \\ y + 3y &= 60 \end{aligned}$$

Solving for x and y yields

$$\begin{aligned} x &= 2z + 60 \\ y &= -3z + 60 \end{aligned}$$

Which corresponds to the ordered triple $(2z + 60, -3z + 60, z)$.

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1. Solve the system of equations below with Gauss-Jordan elimination.

$$\begin{aligned} x + \quad y - \quad z &= 10 \\ 2x - \quad y - \quad z &= 3 \\ -3x - 3y + 3z &= -30 \end{aligned}$$

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Question 5 - How do you mix different grades of ethanol to create a new grade of ethanol?

<u>Key Terms</u>

<u>Summary</u>

In this question, we look at examples of system of equations that come from real problems. In each case, we follow several basic steps to write out the system of equations. Once we have the system, convert the augmented matrix to reduced row echelon form using the rref command on a graphing calculator or Gauss-Jordan Elimination.

The basic steps for solving an application problem are

1. Identify what you are looking for from the problem statement. Assign variables to what you are looking for and describe the variables in detail.
2. Identify key information in the problem statement that relates the variables.
3. Write out equations that correspond to the key information.
4. Solve the system with an appropriate technique.

Let's demonstrate these steps with an example.

> The Riddler Rent-A-Truck company plans to spend \$7 million on 200 new vehicles. Each commercial van will cost \$35,000, each small truck \$30,000, and each large truck \$50,000. They need twice as many vans as small trucks. How many of each vehicle can they buy?

Start by defining the variables:

> V: number of commercial vans to buy
>
> S: number of small trucks to buy
>
> L: number of large trucks to buy

Now let's look at the key information and the corresponding equations:

> **buy 200 new vehicles** $\rightarrow \quad V + S + L = 200$
>
> **spend 7 million** $\rightarrow \quad 35000V + 30000S + 50000L = 7000000$
>
> **need twice as many vans as small trucks** $\rightarrow \quad V = 2S$

The system we need to solve is

$$\begin{aligned} V + S + L &= 200 \\ 35000V + 30000S + 50000L &= 7000000 \\ V &= 2S \end{aligned}$$

Rewriting in the proper form, we get

$$\begin{aligned} V + S + L &= 200 \\ 35000V + 30000S + 50000L &= 7000000 \\ V - 2S &= 0 \end{aligned}$$

The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 200 \\ 35000 & 30000 & 50000 & 7000000 \\ 1 & -2 & 0 & 0 \end{array}\right]$$

Use a graphing calculator or Gauss-Jordan Elimination to get the reduced row echelon form,

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 120 \\ 0 & 1 & 0 & 60 \\ 0 & 0 & 1 & 20 \end{array}\right]$$

or 120 vans, 60 small trucks and 20 large trucks. Note that these numbers match the key information in the problem:

buy 200 new vehicles $\rightarrow$ $120 + 60 + 20 = 200$

spend 7 million $\rightarrow$ $35000(120) + 30000(60) + 50000(20) = 7000000$

need twice as many vans as small trucks→ 120 is twice as big as 60

<u>Notes</u>

Guided Example	Practice
In December of 2014, Sony released the movie *The Interview* online after threats to theaters cancelled the debut in theaters. As originally reported in Wall Street Journal, Sony reported sales of $31 million from the sales and rentals of *The Interview*. They sold the movies online for $15 and rented through various sites for $6. If there were 4.3 million transactions, how many of the transaction were sales of the movie and how many of the transactions were rentals? **Solution** Let's start by defining two variables to what we are looking for: S: number of sales transactions for *The Interview* in millions R: number of rental transactions for *The Interview* in millions These variables also relate to the total sales as well as the total number of transactions. Since there were a total of 4.3 million transactions, $$S+R=4.3$$ Each sale of the movie yields $15 in sales and each rental results in $6 in sales. Thus, the total sales yields $$15S+6R=31$$ Putting these equations together gives the system of linear equations, $$\begin{aligned} S+\ \ R&=4.3 \\ 15S+6R&=31 \end{aligned}$$ Using an augmented matrix, we can write the system as	1. Katherine Chong invests $10,000 received from her grandmother in three ways. With one part, she buys US savings bonds at an interest rate of 2.5% per year. She uses the second part, which amounts to twice the first, to buy mutual funds that offer a return of 6% per year. She puts the rest of the money into a money market account paying 4.5% annual interest. The first year her investments bring a return of $470. How much did she invest in each way?

$$\left[\begin{array}{cc|c} 1 & 1 & 4.3 \\ 15 & 6 & 31 \end{array}\right]$$

Well put this into reduced row echelon form using row operations. The entry in the first row, first column is a 1, so we put a 0 in the second row, first column:

$$-15R_1 + R_2 \to R_2 \quad \left[\begin{array}{cc|c} 1 & 1 & 4.3 \\ 0 & -9 & -33.5 \end{array}\right]$$

Now put a 1 in the second row, second column:

$$-\tfrac{1}{9}R_2 \to R_2 \quad \left[\begin{array}{cc|c} 1 & 1 & 4.3 \\ 0 & 1 & \frac{33.5}{9} \end{array}\right]$$

Finally, put a zero in the first row, second column:

$$-R_2 + R_1 \to R_1 \quad \left[\begin{array}{cc|c} 1 & 0 & \frac{26}{45} \\ 0 & 1 & \frac{33.5}{9} \end{array}\right]$$

This gives the solution $S = \frac{26}{45} \approx 0.58$ and $R = \frac{33.5}{9} \approx 3.72$. Remember that each variable is in millions so there were approximately 0.58 million sale transactions and 3.72 million rental transactions. Based on the critical reviews of the movies as well as the movie *Pineapple Express*, the large number of rentals was justified.