Section 3.2 Matrix Multiplication

Question 1 – How do you multiply two matrices?

Question 2 – How do you interpret the entries in a product of two matrices?

Question 1 – How do you multiply two matrices?

<u>Key Terms</u>

Matrix product

<u>Summary</u>

Suppose we have a 1 x k matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$$

and a k x 1 matrix,

$$B = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix}$$

In each matrix, the dots help to indicate the arbitrary number of rows or columns in each matrix. Although this number k is arbitrary, the number of columns in A must match the number of rows in B. Otherwise it is not possible to carry out the multiplication process.

To find the product these matrices, we must multiply the entries in the row matrix by the entries in the column matrix and add the resulting products:

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix}$$

$$= a_{11} b_{11} + a_{12} b_{21} + \cdots + a_{1n} b_{n1}$$

Notice that each product comes from corresponding columns and rows. In other words, the first product is formed from the first column in the first matrix and the first row in the second matrix, the second product is formed from the second column in the first matrix and the second column in the second matrix, and so on.

Let's try the following product:

$$\begin{bmatrix} 3 & 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -9 \\ -2 \\ 1 \end{bmatrix}$$

To help identify the factors in the products, let's color code each corresponding factor and carry out the sum:

$$\begin{bmatrix} 3 & 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -9 \\ -2 \\ 1 \end{bmatrix} = 3(2) + 1(-9) + (-2)(-2) + 4(1) = 5$$

The key to carrying out the process is to correspond the factors in each product correctly.

How to Multiply Two Matrices

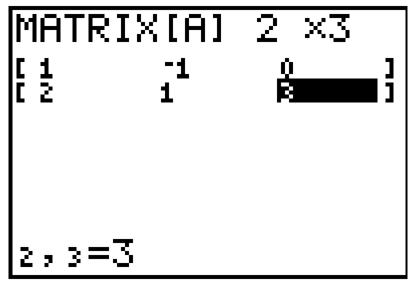
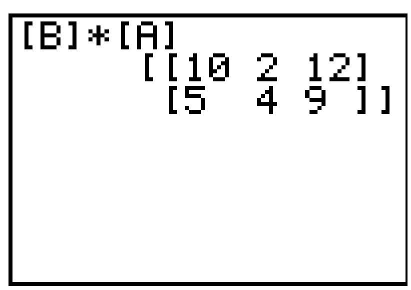
1. Make sure the number of columns in the first matrix matches the number of rows in the second column. If they do not match, the product is not possible.
2. The size of the products is the number of rows in the first matrix by the number of columns in the second matrix. The product of m x k matrix and a k x n matrix is an m x n matrix. Form a matrix of the proper size with blank spaces for each entry.
3. For each entry in the product, form the corresponding factors and sums. The entry in the ith row and jth column of the product is found by corresponding and multiplying the ith row in the first matrix with the jth column in the second matrix.

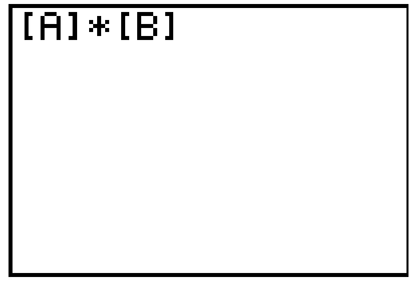
If an m × k matrix is multiplied by a k × n matrix, the product will be a m × n.

$$(m \times k) \quad (k \times n) = m \times n$$

As noted earlier, the number od columns in the first matrix must match the number of rows in the second matrix.

Let’s look at how we can carry out matrix multiplication with technology. For the matrices $A=\begin{bmatrix}1 & -1 & 0\\ 2 & 1 & 3\end{bmatrix}$ and $B=\begin{bmatrix}2 & 4\\ -1 & 3\end{bmatrix}$, find the product BA using a TI graphing calculator.

Enter the matrices 1. Press ψ □ to access the MATRIX menu. 2. Use the right arrow to go to EDIT. 3. Move the cursor to 1: [A] and press ⊆. Note that if you used this matrix name before, it will have a dimension next to it. 4. Enter the dimension of matrix A as 2 x 3. Enter the values into the matrix as shown. Note that the position is given at the bottom of the screen as 2,3=3 etc.	
5. Repeat the process to enter matrix B, i.e. Press ψ □ to return to the MATRIX menu etc. 6. Press ψ ζ to QUIT and return to the Home screen.	
Perform the matrix multiplication on the home screen 7. Use ψ□ to access NAMES. Enter the names of the matrices. You may use the multiplication key to multiply the	

matrices as shown, but it not necessary. 8. Press ⊆ to see the product matrix.	
9. Note that you get a dimension error when you try to do [A] [B]. This is because the number of columns in [A] does not match the number of rows in [B].	 

<u>Notes</u>

Guided Example	Practice
Suppose $A=\begin{bmatrix}1 & 3\\ -1 & 2\end{bmatrix}$ and $B=\begin{bmatrix}10\\ 5\end{bmatrix}$. a. Find AB. **Solution** Matrix A is a 2 × 2 matrix and B is a 2 × 1 matrix. The product will be a 2 × 1 matrix. $AB=\begin{bmatrix}1 & 3\\ -1 & 2\end{bmatrix}\begin{bmatrix}10\\ 5\end{bmatrix}$ $=\begin{bmatrix}1\cdot 10+3\cdot 5\\ -1\cdot 10+2\cdot 5\end{bmatrix}$ $=\begin{bmatrix}25\\ 0\end{bmatrix}$ b. Find BA. **Solution** For the product BA to make sense, the number of columns in B, 1, must match the number of rows in A, 2. Since these do not match, it is not possible to compute the product.	1. Suppose $A=\begin{bmatrix}3\\ 4\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\ -2 & 3\end{bmatrix}$. a. Find AB. b. Find BA.

Guided Example	Practice
Suppose $A=\begin{bmatrix}0 & 5 & -1\\ 2 & 1 & 4\end{bmatrix}$ and $B=\begin{bmatrix}4 & -1\\ 2 & 10\end{bmatrix}$. a. Find *AB*. **Solution** For the product *AB* to make sense, the number of columns in *A*, 3, must match the number of rows in *B*, 2. Since these do not match, it is not possible to compute the product. b. Find *BA*. **Solution** Since the number of columns in *B* matches the number of rows in *A*, it is possible to compute the product. In this problem, we are multiplying a 2 × 2 by a 2 × 3 which results in a 2 × 3. $BA=\begin{bmatrix}4 & -1\\ 2 & 10\end{bmatrix}\begin{bmatrix}0 & 5 & -1\\ 2 & 1 & 4\end{bmatrix}$ $=\begin{bmatrix}4\cdot 0+(-1)2 & 4\cdot 5+(-1)(1) & 4(-1)+(-1)4\\ 2\cdot 0+10\cdot 2 & 2\cdot 5+10\cdot 1 & 2(-1)+10\cdot 4\end{bmatrix}$ $=\begin{bmatrix}-2 & 19 & -8\\ 20 & 20 & 38\end{bmatrix}$	Suppose $A=\begin{bmatrix}1 & 3\\ 2 & 0\\ -1 & 5\end{bmatrix}$ and $B=\begin{bmatrix}6 & 3\\ -1 & 2\end{bmatrix}$. a. Find *AB*. b. Find *BA*.

Question 2 – How do you interpret the entries in a product of two matrices?

Key Terms

Summary

Let's look at an example of how to apply matrix multiplication.

> The Mundo Candy Company makes three types of chocolate candy: Cheery Cherry, Mucho Mocha, and Almond Delight. The company produces its products in San Diego, Mexico City, and Managua using two main ingredients: chocolate and sugar.
>
> Each kilogram of Cheery Cherry requires .5 kg of sugar and .2 kg of chocolate, each kilogram of Mucho Mocha requires .4 kg of sugar and .3 kg of chocolate; and each kilogram of Almond Delight requires .3 kg of sugar and .3 kg of chocolate. The cost of 1 kg of sugar is \$4 in San Diego, \$2 in Mexico City, and \$1 in Managua. The cost of 1 kg of chocolate is \$3 in San Diego, \$5 in Mexico City, and \$7 in Managua.
>
> Put the information above in a matrix in such a way that when you multiply the matrices, you get a matrix representing the ingredient cost of producing each type of candy in each city.

Start by putting the information in a matrix. There are two ingredients and three types of candy so we need either a 2 x 3 or 3 x 2. Either will be fine as long as we label the rows and columns. I choose to use a 2 x 3:

$$\begin{array}{c} \\ \text{sugar} \\ \text{choco} \end{array} \begin{array}{c} \begin{array}{ccc} \text{CC} & \text{MM} & \text{AD} \end{array} \\ \begin{bmatrix} .5 & .4 & .3 \\ .2 & .3 & .3 \end{bmatrix} \end{array}$$

Because the product has to correspond to candy type and cities, the product must be a 3 x 3 matrix. To get this from the 2 x 3 above, we'll need to multiply a 3 x 2 times the 2 x 3. Based on the information above, the rows must correspond to cities and the columns to ingredients:

$$\begin{array}{c} \\ \text{SD} \\ \text{MC} \\ \text{M} \end{array} \begin{array}{c} \begin{array}{cc} \text{sugar} & \text{choco} \end{array} \\ \begin{bmatrix} 4 & 3 \\ 2 & 5 \\ 1 & 7 \end{bmatrix} \end{array}$$

Now let's carry out the multiplication:

$$\begin{bmatrix} 4 & 3 \\ 2 & 5 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} .5 & .4 & .3 \\ .2 & .3 & .3 \end{bmatrix} = \begin{bmatrix} 2.6 & 2.5 & 2.1 \\ 2 & 2.3 & 2.1 \\ 1.9 & 2.3 & 2.4 \end{bmatrix}$$

To get the entry in the second row, first column of the product we need to multiply the second row in the first matrix by the first column in the second matrix and add the results:

$$2(.5)+5(.2)=2$$

Other entries are calculated similarly. Since we are multiplying amounts of ingredients times cost per amount, the product is a total cost. How should we label the product?

$$(\text{city}\times\text{ingredient})(\text{ingredient}\times\text{candy})=\text{city}\times\text{candy}$$

so

$$\begin{array}{c} \\ \text{SD} \\ \text{MC} \\ \text{M} \end{array}\begin{array}{c} \begin{array}{ccc} \text{CC} & \text{MM} & \text{AD} \end{array} \\ \begin{bmatrix} 2.6 & 2.5 & 2.1 \\ 2 & 2.3 & 2.1 \\ 1.9 & 2.5 & 2.4 \end{bmatrix} \end{array}$$

The cost of Cheery Cherry is Mexico City would be $2.

<u>Notes</u>

Guided Example

A political candidate plans to use three methods of advertising: newspapers, radio, and cable TV. The cost per ad (in thousands of dollars) for each type of media is given by matrix A.

Unit Cost

$$A = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} \begin{matrix} \text{news} \\ \text{radio} \\ \text{cable} \end{matrix}$$

Matrix B shows the number of ads per month in these three media that are targeted to single people, to married males aged 35 to 55, and to married females over 65 years of age.

$$\begin{matrix} & \text{news} & \text{radio} & \text{cable} \end{matrix}$$

$$B = \begin{bmatrix} 10 & 15 & 20 \\ 20 & 5 & 10 \\ 5 & 10 & 25 \end{bmatrix} \begin{matrix} \text{single} \\ \text{married men} \\ \text{married female} \end{matrix}$$

Find the matrix that gives the cost of ads for each target group.

Solution The matrix A is a 3 × 1 matrix (media × unit cost) and matrix B is a 3 × 3 matrix (targeted group × media). If we multiply B times A, the number of columns in B (3 representing media) matches the number of rows in A (also representing media).

(targeted group × media)(media × unit cost)

The result will be a 3 × 1 matrix representing (targeted group × cost).

$$BA = \begin{bmatrix} 10 & 15 & 20 \\ 20 & 5 & 10 \\ 5 & 10 & 25 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \cdot 3 + 15 \cdot 1 + 20 \cdot 5 \\ 20 \cdot 3 + 5 \cdot 1 + 10 \cdot 5 \\ 5 \cdot 3 + 10 \cdot 1 + 25 \cdot 5 \end{bmatrix}$$

$$= \begin{bmatrix} 145 \\ 115 \\ 150 \end{bmatrix}$$

Labeling the matrix, we get

$$BA = \begin{bmatrix} 145 \\ 115 \\ 150 \end{bmatrix} \begin{matrix} \text{single} \\ \text{married men} \\ \text{married female} \end{matrix}$$

(column: Cost)

Practice

1. Men and women in a church choir wear choir robes in the sizes shown in matrix A.

$$\begin{matrix} & \text{S} & \text{M} & \text{L} & \text{XL} \end{matrix}$$
$$A = \begin{bmatrix} 8 & 15 & 5 & 0 \\ 1 & 10 & 10 & 5 \end{bmatrix} \begin{matrix} \text{Women} \\ \text{Men} \end{matrix}$$

Matrix B contains the prices (in dollars) of new robes and hoods according to size.

$$\begin{matrix} \text{robes} & \text{hoods} \end{matrix}$$
$$B = \begin{bmatrix} 55 & 25 \\ 65 & 25 \\ 75 & 35 \\ 95 & 35 \end{bmatrix} \begin{matrix} \text{S} \\ \text{M} \\ \text{L} \\ \text{XL} \end{matrix}$$

Find the matrix that gives the total cost of robes and hoods for men and women.