# Section 3.3 Matrix Inverses

Question 1 – What is a matrix inverse?

Question 2 – How do you find a matrix inverse?

## Question 1 – What is a matrix inverse?

Key Terms

| | |
|---|---|
| Square matrix | Identity matrix |
| Inverse of a matrix | Invertible |

Summary

A square matrix is a matrix in which the number of rows is equal to the number of columns.

An identity matrix is a square matrix with ones along the diagonal entries and zeros elsewhere. The letter I is used to denote an identity matrix. If the context of the problem is not clear enough to establish the size of the identity matrix, a subscript is used to establish the size. For instance, $I_2$ would be a $2 \times 2$ identity matrix and $I_3$ would be a $3 \times 3$ identity matrix.

Two square matrices $A$ and $B$ are called inverses if and only if their product is the identity matrix,

$$AB = BA = I$$

The matrix B is called the inverse of A and is written $A^{-1}$. A matrix that has an inverse is called an invertible matrix.

Notes

| Guided Example | Practice |
| --- | --- |
| Are A and B inverses of each other?<br><br>$A=\begin{bmatrix}3 & 5\\ 1 & 2\end{bmatrix}\quad B=\begin{bmatrix}2 & -5\\ -1 & 3\end{bmatrix}$<br><br>**Solution** Matrices that are inverses of each other yield the identity matrix when they are multiplied. Start by multiplying *AB*:<br><br>$AB=\begin{bmatrix}3 & 5\\ 1 & 2\end{bmatrix}\begin{bmatrix}2 & -5\\ -1 & 3\end{bmatrix}$<br>$=\begin{bmatrix}3\cdot 2+5(-1) & 3(-5)+5\cdot 3\\ 1\cdot 2+2(-1) & 1(-5)+2\cdot 3\end{bmatrix}$<br>$=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$<br><br>Now check *BA*:<br><br>$BA=\begin{bmatrix}2 & -5\\ -1 & 3\end{bmatrix}\begin{bmatrix}3 & 5\\ 1 & 2\end{bmatrix}$<br>$=\begin{bmatrix}2\cdot 3+(-5)(1) & 2\cdot 5+(-5)2\\ -1\cdot 3+3\cdot 1 & -1\cdot 5+3\cdot 2\end{bmatrix}$<br>$=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$<br><br>Since *AB* = *BA* = *I*, the matrices are inverses. | 1. Are A and B inverses of each other?<br><br>$A=\begin{bmatrix}5 & 7\\ 3 & 4\end{bmatrix}\quad B=\begin{bmatrix}-4 & 7\\ 3 & -5\end{bmatrix}$ |

<u>Guided Example</u>

If $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -3 \\ -1 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -2 & -1 \\ -1 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix}$, are $A$ and $B$ inverses?

**Solution** Compute the products $AB$ and $BA$ to determine if they are equal to the identity matrix. In both cases, the product of two 3 x 3 matrices is another 3 x 3 matrix.

$$AB = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -3 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 4 & -2 & -1 \\ -1 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 4 + 1(-1) + (-1)(2) & 1(-2) + 1\cdot 1 + (-1)(-1) & 1(-1) + 1\cdot 1 + (-1)(0) \\ 2\cdot 4 + 2(-1) + (-3)(2) & 2(-2) + 2\cdot 1 + (-3)(-1) & 2(-1) + 2\cdot 1 + (-3)(0) \\ (-1)(4) + 0(-1) + 2\cdot 2 & (-1)(-2) + 0\cdot 1 + 2(-1) & (-1)(-1) + 0\cdot 1 + 2\cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 4 & -2 & -1 \\ -1 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -3 \\ -1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4\cdot 1 + (-2)(2) + (-1)(-1) & 4\cdot 1 + (-2)(2) + (-1)(0) & 4(-1) + (-2)(-3) + (-1)(2) \\ (-1)(1) + 1\cdot 2 + 1(-1) & (-1)(1) + 1\cdot 2 + 1\cdot 0 & (-1)(-1) + 1(-3) + 1\cdot 2 \\ 2\cdot 1 + (-1)(2) + 0(-1) & 2\cdot 1 + (-1)(2) + 0\cdot 0 & 2(-1) + (-1)(-3) + 0\cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Both products are equal to the 3 x 3 identity matrix. so

$$A^{-1} = \begin{bmatrix} 4 & -2 & -1 \\ -1 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix}$$

Practice

2. If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, are $A$ and $B$ inverses?

Question 2 – How do you find a matrix inverse?

Key Terms

Summary

To find the inverse of a square matrix, the matrix is combined with an identity matrix of the same size in a single matrix. If the matrix is called *A*, we write this symbolically as $\left[A \mid I\right]$. If the matrix A is a 2 x 2 matrix, combining it with a 2 x 2 identity matrix results in a 2 x 4 matrix.

The inverse matrix is found by using row operations to transform the matrix so that the identity matrix is on the left side of the combined matrix. The right side of the matrix is the inverse of the matrix *A*. Symbolically, we must use row operations to yield $\left[I \mid A^{-1}\right]$.

**How to Find the Inverse of a Matrix**

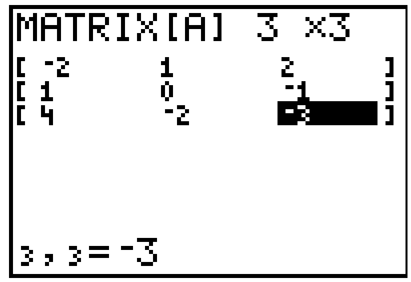
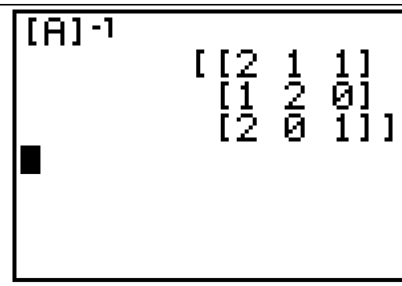
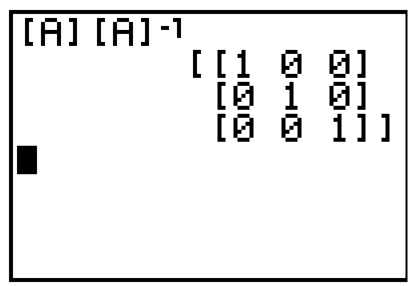
To compute $A^{-1}$ for an *n* x *n* matrix *A*,

1. Place the matrix *A* alongside the identity matrix $I_n$ to form a new matrix $\left[A \mid I_n\right]$
2. Use row operations to place a one in the first row, first column of the matrix.
3. Use row operations to place zeros in the rest of the first column.
4. Continue using row operations to place a one in each column in the row that matches the column number. Once the one is in place in a column, use row operations to make the rest of the entries in that column equal to zero.
5. When the left-hand side of the matrix is equal to $I_n$, the right hand side of the matrix is the inverse of *A* or $A^{-1}$.
6. If any of the rows on the left-hand side of the matrix consists entirely of zeros, then the matrix *A* does not have an inverse. When a matrix does not have an inverse, we say it is not invertible.

A graphing calculator may be used to find the inverse of a matrix. Suppose we start with the matrix

$$A = \begin{bmatrix} -2 & 1 & 2 \\ 1 & 0 & -1 \\ 4 & -2 & -3 \end{bmatrix}.$$

To calculate the inverse of $A$, follow the instructions below.

<table>
<tr><td>Enter the matrix<br><br>Press ψ □ to access the MATRIX menu.<br><br>Move the cursor to 1: [A] and press ⊆.<br><br>Enter the dimension of matrix A as 3 x 3. Enter the values into the matrix as shown. Note that the position is given at the bottom of the screen as 3,3= -3 etc.<br><br>Press ψ ζ to QUIT and return to the Home screen.</td><td>
</td></tr>
<tr><td>Find the matrix inverse on the home screen<br><br>Press ψ□ to enter the name of the matrix. Press 1 or ⊆ to paste [A] to the home screen.<br><br>Press the □ key to find the inverse of matrix A.<br><br>Press ⊆ to see $A^{-1}$.</td><td>
</td></tr>
<tr><td>Show that $AA^{-1} = I$<br><br>On the home screen use ψ⊆ to paste the name of the matrix A.<br><br>On the same line, enter matrix A again, then press the □ key.<br><br>Press ⊆ to see to see the product matrix, the identity matrix for a 3x3 matrix.</td><td>
</td></tr>
</table>

Notes

Guided Example

Find the inverse of the matrix $A = \begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix}$

Solution Start by augmenting the matrix A with a 2 × 2 identity matrix: $\left[\begin{array}{cc|cc} -1 & -2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$.

Now use row operations to transform the matrix to $\left[ I \mid A^{-1} \right]$.

$$-1R_1 \to R_1 \qquad \left[\begin{array}{cc|cc} 1 & 2 & -1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$$

$$-3R_1 + R_2 \to R_2 \qquad \left[\begin{array}{cc|cc} 1 & 2 & -1 & 0 \\ 0 & -2 & 3 & 1 \end{array}\right]$$

$$-\tfrac{1}{2}R_2 \to R_2 \qquad \left[\begin{array}{cc|cc} 1 & 2 & -1 & 0 \\ 0 & 1 & -\tfrac{3}{2} & -\tfrac{1}{2} \end{array}\right]$$

$$-2R_2 + R_1 \to R_1 \qquad \left[\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & 1 & -\tfrac{3}{2} & -\tfrac{1}{2} \end{array}\right]$$

The right-hand side of the matrix is $A^{-1}$,

$$A^{-1} = \begin{bmatrix} 2 & 1 \\ -\tfrac{3}{2} & -\tfrac{1}{2} \end{bmatrix}$$

<u>Practice</u>

1. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

<u>Guided Example</u>

Find the inverse of the matrix $A=\begin{bmatrix}1 & 1 & -1\\ 2 & 1 & 1\\ 1 & 0 & 1\end{bmatrix}$.

**Solution** Form the initial matrix $\left[\begin{array}{ccc|ccc}1 & 1 & -1 & 1 & 0 & 0\\ 2 & 1 & 1 & 0 & 1 & 0\\ 1 & 0 & 1 & 0 & 0 & 1\end{array}\right]$. Now carry out row operation to but a 3 × 3 identity matrix on the left side of this matrix.

$$\begin{array}{l}-2R_1+R_2\to R_2\\ -1R_1+R_3\to R_3\end{array}\qquad \left[\begin{array}{ccc|ccc}1 & 1 & -1 & 1 & 0 & 0\\ 0 & -1 & 3 & -2 & 1 & 0\\ 0 & -1 & 2 & -1 & 0 & 1\end{array}\right]$$

$$-1R_2\to R_2\qquad \left[\begin{array}{ccc|ccc}1 & 1 & -1 & 1 & 0 & 0\\ 0 & 1 & -3 & 2 & -1 & 0\\ 0 & -1 & 2 & -1 & 0 & 1\end{array}\right]$$

$$\begin{array}{l}-1R_2+R_1\to R_1\\ R_2+R_3\to R_3\end{array}\qquad \left[\begin{array}{ccc|ccc}1 & 0 & 2 & -1 & 1 & 0\\ 0 & 1 & -3 & 2 & -1 & 0\\ 0 & 0 & -1 & 1 & -1 & 1\end{array}\right]$$

$$-1R_3\to R_3\qquad \left[\begin{array}{ccc|ccc}1 & 0 & 2 & -1 & 1 & 0\\ 0 & 1 & -3 & 2 & -1 & 0\\ 0 & 0 & 1 & -1 & 1 & -1\end{array}\right]$$

$$\begin{array}{l}3R_3+R_2\to R_2\\ -2R_3+R_1\to R_1\end{array}\qquad \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & -1 & 2\\ 0 & 1 & 0 & -1 & 2 & -3\\ 0 & 0 & 1 & -1 & 1 & -1\end{array}\right]$$

So the inverse matrix is

$$A^{-1}=\begin{bmatrix}1 & -1 & 2\\ -1 & 2 & -3\\ -1 & 1 & -1\end{bmatrix}$$

Practice

2. Find the inverse of the matrix $A=\begin{bmatrix}1 & 1 & 1\\ 2 & 1 & 1\\ 1 & 1 & 2\end{bmatrix}$.