Section 4.3 The Simplex Method and the Standard Maximization Problem

Question 1 – What is a standard maximization problem?

Question 2 – What are slack variables?

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Question 4 - How do you get the optimal solution to a standard maximization problem with the Simplex Method?

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Question 1 – What is a standard maximization problem?

Key Terms

Standard maximization problem

Summary

A standard maximization problem is a type of linear programming problem in which the objective function is to be maximized and has the form

$$z = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

where $a_1, \ldots, a_n$ are real numbers and $x_1, \ldots, x_n$ are decision variables. The decision variables must represent non-negative values. The other constraints for the standard maximization problem have the form

$$b_1x_1 + b_2x_2 + \cdots + b_nx_n \le c$$

where $b_1, \ldots, b_n$ and c are real numbers and $c \ge 0$.

The variables may have different names, but in standard maximization problems four elements must be present:

1. The objective function is maximized.
2. The objective function must be linear.
3. The constraints are linear where the variables are less than or equal to a nonnegative constant.
4. The decision variables must be nonnegative.

Maximize $P = 100x_1 + 80x_2$

subject to

$$x_1 + x_2 \le 50{,}000$$
$$69.75x_1 + 85.25x_2 \le 4{,}000{,}000$$
$$23.8x_1 + 10.85x_2 \le 1{,}000{,}000$$
$$x_1 \ge 0,\ x_2 \ge 0$$

The objective function is maximized

The objective function has the form $z = a_1x_1 + a_2x_2$

All constraints have the form $b_1x_1 + b_2x_2 \le c$ where c is nonnegative.

Decision variables are nonnegative

<u>Notes</u>

<u>Guided Example</u> <u>Practice</u>

Is the linear programming problem

$$\text{Maximize } z = 5x_1 + 6x_2$$

subject to

$$2x_1 + x_2 \le 4$$

$$x_1 + 2x_2 \le 4$$

$$x_1 \ge 0,\ x_2 \ge 0$$

a standard maximization problem?

Solution To see whether this linear programing problem is a standard linear programming problem, check the requirements above.

Maximize → The objective function is maximized

$z = 5x_1 + 6x_2$ → The objective function has the form $z = a_1x_1 + a_2x_2$

subject to

$2x_1 + x_2 \le 4$

$x_1 + 2x_2 \le 4$

→ All constraints have the form $b_1x_1 + b_2x_2 \le c$ where c is nonnegative.

$x_1 \ge 0,\ x_2 \ge 0$ → Decision variables are nonnegative

Since all the requirements are met, this is a standard minimization problem.

1. Is the linear programming problem

$$\text{Maximize } z = 3x_1 + 4x_2$$

subject to

$$x_1 + x_2 \le 40$$

$$x_1 + 2x_2 \le 60$$

$$x_1 \ge 0,\ x_2 \ge 0$$

a standard maximization problem?

Question 2 – What are slack variables?

Key Terms

Slack variables	Initial simplex tableau
Initial simplex tableau	Indicator row

Summary

Slack variables are extra variables that are nonnegative that are added to constraints to change them from inequalities to equalities. For instance, in the standard maximization problem below,

$$\text{Maximize} \quad z = 3x_1 + 4x_2 \text{ subject to}$$

$$2x_1 + 3x_2 \le 6$$

$$2x_1 + x_2 \le 4$$

$$x_1 \ge 0, x_2 \ge 0$$

The constraints are

$$2x_1 + 3x_2 \le 6$$

$$2x_1 + x_2 \le 4$$

We can change these inequalities to equalities by adding a nonnegative number to the left side of each inequality. If the slack variables are called s_1 and s_2, then the inequalities become

$$2x_1 + 3x_2 + s_1 = 6$$

$$2x_1 + x_2 + s_2 = 4$$

The objective function $z = 3x_1 + 4x_2$ is already an equation, but we can write it with all of the variables on the left side as

$$-3x_1 - 4x_2 + z = 0$$

Put all of these into a system of linear equations and we get

$$\begin{aligned} 2x_1 + 3x_2 + s_1 \qquad\quad &= 6 \\ 2x_1 + x_2 \qquad + s_2 \quad &= 4 \\ -3x_1 - 4x_2 \qquad\quad + z &= 0 \end{aligned}$$

This is a system of three linear equations in five variables. The corresponding augmented matrix is

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ \hline 2 & 3 & 1 & 0 & 0 & 6 \\ 2 & 1 & 0 & 1 & 0 & 4 \\ \hline -3 & -4 & 0 & 0 & 1 & 0 \end{array}$$

This matrix is called the initial simplex tableau. The bottom row in the tableau always originates from the objective function and is called the indicator row.

Notes

Guided Example	Practice
Find the initial simplex tableau for the linear programming problem. Maximize $z = 2x_1 + 4x_2$ subject to $x_1 + x_2 \le 10$ $15x_1 + 10x_2 \le 120$ $x_1 \ge 0,\ x_2 \ge 0$ **Solution** Add slack variables to the constraints to give $x_1 + x_2 + s_1 = 10$ $15x_1 + 10x_2 + s_2 = 120$	1. Find the initial simplex tableau for the linear programming problem. Maximize $z = 3x_1 + 4x_2$ subject to $x_1 + x_2 \le 40$ $x_1 + 2x_2 \le 60$ $x_1 \ge 0,\ x_2 \ge 0$

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The objective function can be rewritten as $-2x_1 - 4x_2 + z = 0$. Putting these equations together gives the system

$$\begin{aligned} x_1 + x_2 + s_1 &= 10 \\ 15x_1 + 10x_2 + s_2 &= 120 \\ -2x_1 - 4x_2 + z &= 0 \end{aligned}$$

Write this system in matrix form to give the initial simplex tableau

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & z \end{matrix} \\ \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 10 \\ 15 & 10 & 0 & 1 & 0 & 120 \\ \hline -2 & -4 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$
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<u>Notes</u>

Question 3 – How do you find a basic feasible solution?

<u>Key Terms</u>

Basic feasible solution Basic variable

Nonbasic variable

<u>Summary</u>

A simplex tableau represents a system of equations in many variables. Typically, there are more variables than equations which means that the system is dependent has an infinite number of solutions. For instance, the initial simplex tableau

$$\begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array}$$
$$\left[\begin{array}{ccccc|c} 2 & 3 & 1 & 0 & 0 & 6 \\ 2 & 1 & 0 & 1 & 0 & 4 \\ \hline -3 & -4 & 0 & 0 & 1 & 0 \end{array}\right]$$

has five variables and three equations. If we were to solve this system, we would be able to solve for three of the variables in terms of the other two variables. For this system, it would be easy to solve for s_1, s_2, and z since the columns corresponding to those variables contain ones and zeros. Because of this, these variables are called basic variables. The other two variables, x_1 and x_2, are called nonbasic variables. The numbers in these columns typically do not consist of ones and zeros.

Any simplex tableau corresponds to a solution that may be found by setting the nonbasic variables equal to zero. This has the effect of eliminating those columns from the system. If we set x_1 and x_2 equal to zero, we can cover up those columns and read a solution from the remaining part of the matrix.

$$\begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array}$$
$$\left[\begin{array}{ccccc|c} 2 & 3 & 1 & 0 & 0 & 6 \\ 2 & 1 & 0 & 1 & 0 & 4 \\ \hline -3 & -4 & 0 & 0 & 1 & 0 \end{array}\right]$$

The basic feasible solution corresponding to this simplex tableau is $x_1 = 0$, $x_2 = 0$, $s_1 = 6$, $s_2 = 4$ and $z = 0$. If we had solved this problem geometrically, this point would have corresponded to the corner point at the origin.

Other basic feasible solutions are obtained by putting ones and zeros in different columns and setting the nonbasic variables equal to zero. This is done by carrying out row operations. For instance, suppose we apply the row operations below on the matrix above:

$$\begin{array}{r} \frac{1}{3}R_1 \to R_1 \\ -1R_1 + R_2 \to R_2 \\ 4R_1 + R_3 \to R_3 \end{array} \quad \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} \frac{2}{3} & 1 & \frac{1}{3} & 0 & 0 & 2 \\ \frac{4}{3} & 0 & -\frac{1}{3} & 1 & 0 & 2 \\ \hline -\frac{1}{3} & 0 & \frac{4}{3} & 0 & 0 & 8 \end{array}\right] \end{array}$$

The solution corresponding to this simplex tableau is $x_1 = 0$, $x_2 = 2$, $s_1 = 0$, $s_2 = 2$ and $z = 8$. This point also corresponds to a corner point on the feasible region.

By selecting different basic and nonbasic variables, we can find every corner point on the feasible region for the linear programming problem.

Notes

Guided Example	Practice
Find the basic feasible solution corresponding to the simplex tableau $$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc\|c} 1 & 1 & 1 & 0 & 0 & 10 \\ 5 & 0 & -10 & 1 & 0 & 20 \\ \hline 2 & 0 & 4 & 0 & 1 & 40 \end{array}\right] \end{array}$$ **Solution** The nonbasic variables are x_1 and s_1. Cover the variables in the simplex tableau: $$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc\|c} 1 & 1 & 1 & 0 & 0 & 10 \\ 5 & 0 & -10 & 1 & 0 & 20 \\ \hline 2 & 0 & 4 & 0 & 1 & 40 \end{array}\right] \end{array}$$ This gives the basic feasible solution $x_1 = 0$, $x_2 = 10$, $s_1 = 0$, $s_2 = 20$ and $z = 40$.	1. Find the basic feasible solution corresponding to the simplex tableau $$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc\|c} 0 & 2 & 1 & -1 & 0 & 40 \\ 1 & 1 & 0 & \frac{1}{2} & 0 & 20 \\ \hline 0 & -1 & 0 & \frac{3}{2} & 1 & 120 \end{array}\right] \end{array}$$

Question 4 – How do you get the optimal solution to a standard maximization problem with the Simplex Method?

Key Terms

Pivot row

Pivot column

Pivot

Summary

The Simplex Method is a technique for discovering which variables should be basic and which should be nonbasic. It is an iterative procedure which will find the solution of any standard minimization problem.

1. Make sure the linear programming problem is a standard maximization problem.

2. Convert each inequality to an equality by adding a slack variable. Each inequality must have a different slack variable. Each constraint will now be an equality of the form

$$b_1x_1 + b_2x_2 + \cdots + b_nx_n + s = c$$

   where s is the slack variable for the constraint. If more than one slack variable is needed, use subscripts like $s_1, s_2, \ldots$

3. Rewrite the objective function $z = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ by moving all of the variables to the left side. After rewriting the equation, the function will have the form

$$-a_1x_1 - a_2x_2 - \cdots - a_nx_n + z = 0.$$

4. Convert the equations from steps 2 and 3 to an initial simplex tableau. Put the equation from step 3 in the bottom row of the tableau and all other equations above it. The bottom row is called the indicator row.

5. Find the entry in the indicator row that is most negative. If two of the entries are most negative and equal, pick the entry that is farthest to the left. The column with this entry is called the pivot column.

6. For each row except the last row, divide the entry in the last column by the entry in the pivot column. Ignore any rows where the entry in the pivot column is negative. The row with the smallest non-negative quotient is the pivot row. If more than one row has the same smallest quotient, the higher of the rows is the pivot row.

7. The pivot is the entry where the pivot row and pivot column intersect. Multiply the pivot row by the reciprocal of the pivot to change it to a 1.

8. To change the rest of the pivot column to zeros, multiply the pivot row by constants and add them to the other rows in the tableau. Replace those rows with the appropriate sums. When complete, the pivot should be a one, and the rest of the pivot column should be zeros.

9. If the indicator does not contain any negative entries, this tableau corresponds to the optimum solution. In this case, cover the nonbasic variables (set the nonbasic variables equal to zero), and read off the solution for the basic variables. Otherwise, repeat steps 5 through 9 until the indicator row contains no negative numbers.

<u>Notes</u>

Guided Example

Find the optimal solution for the linear programming problem below using the simplex method.

Maximize $z = 60x_1 + 50x_2$

subject to

$$x_1 + x_2 \le 100$$

$$x_1 + 2x_2 \le 180$$

$$x_1 \ge 0,\ x_2 \ge 0$$

Solution Start by adding slack variables to each inequality to change it into an equation:

$$x_1 + x_2 + s_1 = 100$$

$$x_1 + 2x_2 + s_2 = 180$$

Rewrite the objective function to put all of its variable terms on the left side of the equation:

$$-60x_1 - 50x_2 + z = 0$$

Put these equations into the initial simplex tableau with the objective function equation in the bottom row:

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ \hline 1 & 1 & 1 & 0 & 0 & 100 \\ 1 & 2 & 0 & 1 & 0 & 180 \\ \hline -60 & -50 & 0 & 0 & 1 & 0 \end{array}$$

The pivot column will be the first column since the most negative indicator is in the first column. The pivot row is the second row since the ratio $\frac{100}{1}$ is smaller than $\frac{180}{1}$. This makes the pivot the number 1 in the first row, first column.

Since the pivot is already a 1 (if it is not divide the row by the pivot to make it a 1), use the row

Practice

1. Find the optimal solution for the linear programming problem below using the simplex method.

Maximize $z = 3x_1 + 4x_2$

subject to

$$x_1 + x_2 \le 40$$

$$x_1 + 2x_2 \le 60$$

$$x_1 \ge 0,\ x_2 \ge 0$$

operations $-1R_1 + R_2 \to R_2$ and $60R_1 + R_3 \to R_3$ to put zeros in the rest of the pivot column:

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & z & \end{matrix} \\ \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 100 \\ 0 & 1 & -1 & 1 & 0 & 80 \\ \hline 0 & 10 & 60 & 0 & 1 & 6000 \end{array}\right] \end{array}$$

The indicator row does not contain any negative indicators, so this tableau is the final tableau. It corresponds to the solution $x_1 = 100$, $x_2 = 0$, and $z = 6000$.

If the indicator row had still contained a negative indicator, we would have picked a new pivot and used row operations to change the pivot to a one. More row operations would then be used to make the other entries in the pivot column into zeros.

Question 5 – How do you find the optimal solution for an application?

<u>Key Terms</u>

<u>Summary</u>

A linear programming application can be broken down into two parts. First, you need to set up the application by writing out the variables, objective function, and constraints. Once the linear programming problem is written down and determined to be a standard maximization problem, we can solve the problem with the Simplex Method.

<u>Notes</u>

Guided Example

Carrie Green is working to raise money for the homeless by sending information letters and making follow up calls to local labor organizations and church groups. She discovers that each church group requires 2 hours of letter writing and 1 hour of follow up while for each labor union she needs 2 hours of letter writing and 3 hours of follow up. Carrie can raise \$100 from each church group and \$200 from each union local, and she has a maximum of 16 hours of letter writing time and a maximum of 12 hours of follow up time available per month. Determine the most profitable mixture of groups she should contact and the most money she can raise.

Solution Follow the steps outlined above.

Set Up The Linear Programming Problem - To get started, we need to identify the variables in this problem. Since the problem asks us to "determine the most profitable mixture of groups she should contact", let's define

C: number of church groups to contact

U: number of union locals to contact

With these two variables defined, let's find the objective function. The problem statement asks us to determine "the most money she can raise". Since she can raise \$100 from each church group and \$200 from each union local, the objective function must be

$$Z = 100C + 200U$$

Now look for the factors that constrain her fundraising. Two pieces of information are evident:

she has a maximum of 16 hours of letter writing time

she has a maximum of 12 hours of follow up time

This leads me to write

$$\text{total amount of letter writing time} \le 16 \text{ hours}$$
$$\text{total amount of follow up time} \le 12 \text{ hours}$$

Let's tackle the first piece of information. Since it regards letter writing, let's find the information for letter writing.

each church group requires 2 hours of letter writing

each labor union she needs 2 hours of letter writing

So, if we have C church groups and U union locals, we can write

$$2C + 2U \leq 16$$

Following a similar strategy for follow up leads us to

$$C + 3U \leq 12$$

Now that we have the objective function and the constraints, we can write out the linear programming problem:

$$\begin{aligned} &\text{Maximize} \quad Z = 100C + 200U \\ &\text{subject to} \\ &\quad 2C + 2U \leq 16 \\ &\quad C + 3U \leq 12 \\ &\quad C \geq 0, U \geq 0 \end{aligned}$$

Now we can carry out the simplex method.

Carry Out The Simplex Method - Rewrite the problem with two slack variables:

$$\begin{aligned} &\text{Maximize} \quad Z = 100C + 200U \\ &\text{subject to} \\ &\quad 2C + 2U + s_1 \qquad = 16 \\ &\quad C + 3U \qquad + s_2 = 12 \\ &\quad C \geq 0, U \geq 0, s_1 \geq 0, s_2 \geq 0 \end{aligned}$$

The initial tableau is

$$\begin{array}{c} \begin{matrix} C & U & s_1 & s_2 & Z \end{matrix} \\ \left[\begin{array}{ccccc|c} 2 & 2 & 1 & 0 & 0 & 16 \\ 1 & 3 & 0 & 1 & 0 & 12 \\ \hline -100 & -200 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The pivot column is the second column since -200 is the most negative entry in the indicator row. The pivot row is the second row since $\frac{12}{3} < \frac{16}{2}$. Therefore, we need to change the 3 in the pivot entry to a 1 by performing $\frac{1}{3}R_2 \to R_2$:

$$\begin{array}{cccccc} C & U & s_1 & s_2 & Z & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 2 & 2 & 1 & 0 & 0 & 16 \\ \frac{1}{3} & 1 & 0 & \frac{1}{3} & 0 & 4 \\ \hline -100 & -200 & 0 & 0 & 1 & 0 \end{array}\right]$$

Now we need to put 0's in the rest of the pivot column by performing $-2R_2 + R_1 \to R_1$ and $200R_2 + R_3 \to R_3$:

$$\begin{array}{cccccc} C & U & s_1 & s_2 & Z & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} \frac{4}{3} & 0 & 1 & -\frac{2}{3} & 0 & 8 \\ \frac{1}{3} & 1 & 0 & \frac{1}{3} & 0 & 4 \\ \hline -\frac{100}{3} & 0 & 0 & \frac{200}{3} & 1 & 800 \end{array}\right]$$

Since there is a negative number in the indicator row, we need to pivot again. The new pivot is the $\frac{4}{3}$ in the first row, first column. We begin by changing the pivot to a 1 by performing $\frac{3}{4}R_1 \to R_1$:

$$\begin{array}{cccccc} C & U & s_1 & s_2 & Z & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 0 & \frac{3}{4} & -\frac{2}{4} & 0 & 6 \\ \frac{1}{3} & 1 & 0 & \frac{1}{3} & 0 & 4 \\ \hline -\frac{100}{3} & 0 & 0 & \frac{200}{3} & 1 & 800 \end{array}\right]$$

To put 0's in the rest of the column, perform $-\frac{1}{3}R_1 + R_2 \to R_2$ and $\frac{100}{3}R_1 + R_3 \to R_3$:

$$\begin{array}{cccccc} C & U & s_1 & s_2 & Z & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 0 & \frac{3}{4} & -\frac{2}{4} & 0 & 6 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & 0 & 2 \\ \hline 0 & 0 & 25 & 50 & 1 & 1000 \end{array}\right]$$

Since there are no negative numbers in the indicator row, we have arrived at the maximum amount of money raised, \$1000. This is done by contacting 6 church groups and 2 union locals.

Practice

A convenience store sells three types of juices: grape, cranberry, and mango. It earns $0.60, $0.76, and $0.99 in profit on each bottle of the three juices, respectively. It can stock no more than 400 bottles in the store each week. Typically, at least twice as many cranberry bottles are sold as mango bottles. The company never sells more than 100 bottles of grape juice in a week. How many bottles of each juice should the store stock to maximize profit?